INFINITE DIMENSIONAL TILTING THEORY AND ITS APPLICATIONS

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Introduction

Classical tilting theory originated in the representation theory of finite dimensional algebras, as one of the basic tools for investigation of finite dimensional modules over algebras close to the hereditary ones. In the more general setting of modules over associative rings, tilting theory is primarily a far reaching generalization of the Morita theory of equivalence of module categories. Though many of the classical results require the tilting modules to be finitely generated, there are some that have extensions to infinite dimensional modules, and provide for unexpected new applications.

Many of these new results are related to approximation theory of modules. This theory attempts to study arbitrary modules by a two step procedure: first, one selects a class of modules $\mathcal{C}$ that is well–understood, and then tries to approximate arbitrary modules by describing their approximations (envelopes or covers) in $\mathcal{C}$. Approximation theory originated in the works of Baer and Eckmann on injective hulls, and Bass on projective covers, but there are much more recent contributions, for example the Enochs and Xu theory of flat covers of modules. In fact, there is now a general theory available giving existence of numerous other approximations arising from so called complete cotorsion pairs.

Tilting approximations in projective dimension one are exactly the special approximations given by torsion classes of modules. Here the representing tilting modules are not finitely generated in general. Indeed, in many cases (e.g., for Dedekind domains), there exist no non–trivial finitely generated tilting modules. Moreover, infinite dimensional tilting modules arise naturally even in the setting of finite dimensional hereditary algebras as shown by the examples of Ringel and Lukas tilting modules.

The case of n–tilting modules and classes is more complex, but even here, there is a hidden finiteness condition: each tilting module is of finite type. In particular, the corresponding tilting class is definable in the language of the first order theory of modules. This result was obtained gradually in a series of recent papers in 2003–6, and enabled explicit classification of tilting modules and classes over many rings.

Infinite dimensional tilting modules have interesting applications. For example, they are employed in the proof of the following result of decomposition theory: given a commutative ring R and a multiplicative set S consisting of (some) non–zero–divisors of R, the localization $S^{-1}R$ has projective dimension ≤ 1 (i.e., $S^{-1}R$ is a Matlis localization), iff the R–module $S^{-1}R/R$ decomposes into a direct sum of countably presented modules.

Infinite dimensional tilting modules also come up naturally in the study of finitistic dimensions of rings and algebras. Using them, one can prove the Bass finitistic dimension conjectures for all artin algebras with $\mathcal{P}^{<\omega}$ contravariantly finite, and for all (non–commutative) Iwanaga–Gorenstein rings.

The lecture notes are divided into five chapters. The first one introduces fundamentals of the approximation theory of modules using complete cotorsion pairs as the basic tool. The second deals with deconstruction of cotorsion pairs, which is a method for proving completeness. Here, some new techniques of set–theoretic homological algebra are presented that are interesting on its own (notably, a general version of the Hill lemma).

The third chapter concerns tilting and cotilting modules and proves the finite type result mentioned above. This result makes it possible to classify tilting and cotilting modules in various cases: here we consider in detail the case of Dedekind domains. The final two chapters are dedicated to applications to the structure of Matlis localizations, and to the proof of the Bass finitistic dimension conjectures in the two cases mentioned above.

The lecture notes are based on several recent papers. The lack of space does not allow for a more complete presentation, though we believe that the main ideas are covered here. For more details, and many other related results, we refer to the forthcoming monograph [46].

1 Approximations and cotorsion pairs

In this chapter, we introduce approximation theory as a tool for studying modules over general associative rings with unit. We also introduce the notion of a cotorsion pair which connects left and right approximations in a natural way. Then we consider existence of minimal approximations, and show that there is always a rich supply of approximations available. All these results will later on serve for the structure theory of infinite dimensional tilting modules, and for its applications.

Let R be a ring, M is a (right R–) module, and $\mathcal{C} \subseteq \text{Mod–}R$ a class of modules closed under isomorphic images and direct summands.

Definition 1.1. A map $f \in \operatorname{Hom}_R(M, C)$ with $C \in \mathcal{C}$ is a *$\mathcal{C}$–preenvelope* of M, provided the abelian group homomorphism $\operatorname{Hom}_R(f, C') : \operatorname{Hom}_R(C, C') \to \operatorname{Hom}_R(M, C')$ is surjective for each $C' \in \mathcal{C}$. That is, for each homomorphism $f' : M \to C'$ there is a homomorphism $g : C \to C'$ such that $f' = gf$:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "f'"'] & C \arrow[d, dashed, "g"] \\ & C'. \end{tikzcd}$$

(Note that we require the existence, but not the uniqueness, of the map g.)
The $\mathcal{C}$–preenvelope f is a *$\mathcal{C}$–envelope* of M provided that f is *left minimal*, that is, provided $f = gf$ implies g is an automorphism for each $g \in \operatorname{End}_R(C)$.

Example 1.2. The embedding $M \hookrightarrow E(M)$ of a module into its injective hull is easily seen to be the $\mathcal{I}_0$–envelope of a module M. Similarly, the embedding $M \hookrightarrow PE(M)$ of a module into its pure–injective hull is the $\mathcal{PI}$–envelope of M. (Here, $\mathcal{PI}$ denotes the class of all pure–injective modules and, for each $n < \omega$, $\mathcal{I}_n$ denotes the class of all modules of injective dimension $\leq n$. Moreover, we define $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n$)

Clearly, a $\mathcal{C}$–envelope of M is unique in the following sense: if $f : M \to C$ and $f' : M \to C'$ are $\mathcal{C}$–envelopes of M, then there is an isomorphism $g : C \to C'$ such that $f' = gf$.

In general a module M may have many non–isomorphic $\mathcal{C}$–preenvelopes, but no $\mathcal{C}$–envelope. Nevertheless, if the $\mathcal{C}$–envelope exists, its minimality implies that it is isomorphic to a direct summand in each $\mathcal{C}$–preenvelope of M:

Lemma 1.3. *Let $f : M \to C$ be a $\mathcal{C}$–envelope and $f' : M \to C'$ a $\mathcal{C}$–preenvelope of a module M. Then*

(a) *$C' = D \oplus D'$, where $\operatorname{Im} f' \subseteq D$ and $f' : M \to D$ is a $\mathcal{C}$–envelope of M;*

(b) *f' is a $\mathcal{C}$–envelope of M, iff C' has no proper direct summands containing $\operatorname{Im} f'$.*

Proof. (a) By definition there are homomorphisms $g : C \to C'$ and $g' : C' \to C$ such that $f' = gf$ and $g'g$ is an automorphism of C. So $D = \operatorname{Im} g \cong C$ is a direct summand in C' containing $\operatorname{Im} f'$, and the assertion follows.
(b) by part (a). ∎

Definition 1.4. A class $\mathcal{C} \subseteq \text{Mod–}R$ is a *preenveloping class* (*enveloping class*) provided that each module has a $\mathcal{C}$–preenvelope ($\mathcal{C}$–envelope).

For example, the classes $\mathcal{I}_0$ and $\mathcal{PI}$ from Example 1.2 are enveloping classes of modules.

It is easy to define the dual notions:

Definition 1.5. A map $f \in \text{Hom}_R(C, M)$ with $C \in \mathcal{C}$ is a *$\mathcal{C}$–precover* of M, provided the group homomorphism $\text{Hom}_R(C', f) : \text{Hom}_R(C', C) \to \text{Hom}_R(C', M)$ is surjective for each $C' \in \mathcal{C}$.
A $\mathcal{C}$–precover $f \in \text{Hom}_R(C, M)$ of M is called a *$\mathcal{C}$–cover*of M, provided that f is *right minimal*, that is, provided $fg = f$ implies that g is an automorphism for each $g \in \text{End}_R(C)$.
$\mathcal{C} \subseteq \text{Mod–}R$ is a *precovering class* (*covering class*) provided that each module has a $\mathcal{C}$–precover ($\mathcal{C}$–cover).

$\mathcal{C}$–preenvelopes and $\mathcal{C}$–precovers are also called *left* and *right approximations*.
If Mod–R is replaced by its subcategory mod–R in the definitions above, then preenveloping and precovering classes are called *covariantly finite* and *contravariantly finite*, respectively.

Example 1.6. Each module M has a $\mathcal{P}_0$–precover (since each module is a homomorphic image of a projective module). Moreover, M has a $\mathcal{P}_0$–cover, iff M has a projective cover in the sense of Bass (that is, there is an epimorphism $f : P \to M$ with P projective and $\text{Ker}(f)$ a small submodule of P). So $\mathcal{P}_0$ is always a precovering class, and it is a covering class, iff R is a right perfect ring. (Here, for each $n < \omega$, $\mathcal{P}_n$ and $\mathcal{F}_n$ denotes the class of all modules of projective and flat dimension $\leq n$, respectively. Moreover, we define $\mathcal{P} = \bigcup_{n<\omega} \mathcal{P}_n$.)

$\mathcal{C}$–covers may not exist in general, but if they exist, they are unique up to isomorphism. As in Lemma 1.3, we get

Lemma 1.7. *Let $f : C \to M$ be the $\mathcal{C}$–cover of M. Let $f' : C' \to M$ be any $\mathcal{C}$–precover of M. Then*

(a) *$C' = D \oplus D'$, where $D \subseteq \text{Ker}\, f'$ and $f' \restriction D'$ is a $\mathcal{C}$–cover of M.*

(b) *f' is a $\mathcal{C}$–cover of M, iff C' has no non–zero direct summands contained in* $\text{Ker}\, f'$.

The following sufficient condition for the existence of minimal approximations is due to Zimmermann:

Lemma 1.8. *Let R be a ring. Let $f \in \text{Hom}_R(M, C)$ be a $\mathcal{C}$–preenvelope of a module M. Let $E = \text{End}_R(C)$ and $I = \{g \in E \mid gf = 0\}$. Assume that idempotents lift modulo* $\text{Rad}(E)$*, and that there exists a left ideal J of E such that $I + J = E$ and $I \cap J \subseteq \text{Rad}(E)$. Then M has a $\mathcal{C}$–envelope.*

Proof. For $x \in E$, put $\bar{x} = x + \text{Rad}(E)$. By assumption there exist $x \in I$ and $y \in J$ such that $\bar{x}$ and $\bar{y}$ are orthogonal idempotents in $E/\text{Rad}(E)$ and $\bar{x} + \bar{y} = \bar{1}$.

By assumption there is an idempotent $e' \in E$ with $\bar{e'} = \bar{y}$. Put $u = 1 - (e' - y)$. Then u is invertible in E and $e'(u - y) = 0$. Moreover, $y' = u^{-1}e'u = u^{-1}e'y \in J$ is an idempotent such that $\bar{y'} = \bar{y}$.

Since $1 - (x + y') \in \mathrm{Rad}(E)$, there is some $v \in E$ with $v(x + y') = 1$. Put $e = y' + (1 - y')vy' \in J$. Then e is an idempotent such that $1 - e \in I$ and $I \cap Ee \subseteq (I \cap J)e \subseteq \mathrm{Rad}(E)e$. In particular, the left annihilator of f in the ring eEe is contained in $\mathrm{Rad}(eEe) = e\mathrm{Rad}(E)e$. Now, if $g \in eEe$ is such that $gf = ef$, then g is invertible in eEe. It follows that $f' = ef \in \mathrm{Hom}_R(M, eC)$ is left minimal. Since $eC \in \mathcal{C}$, we conclude that f' is a $\mathcal{C}$–envelope of M. ∎

Dually, one can prove existence of a $\mathcal{C}$–cover assuming that idempotents lift modulo $\mathrm{Rad}(E)$, and there exists a right ideal J of E such that $I + J = E$ and $I \cap J \subseteq \mathrm{Rad}(E)$ where $E = \mathrm{End}_R(C)$, $I = \{g \in E \mid fg = 0\}$, and $f \in \mathrm{Hom}_R(C, M)$ is a $\mathcal{C}$–precover.

In particular, minimal versions of approximations in classes of finite length modules always exist:

Corollary 1.9. *Assume M has a $\mathcal{C}$–preenvelope, $f \in \mathrm{Hom}_R(M, C)$, such that $\mathrm{End}_R(C)$ is a semiperfect ring (for example, assume that C has finite length). Then M has a $\mathcal{C}$–envelope.*

Proof. By Lemma 1.8. ∎

Similarly, M has a $\mathcal{C}$–cover provided M has a $\mathcal{C}$–precover, $f \in \mathrm{Hom}_R(M, C)$, such that $\mathrm{End}_R(C)$ is semiperfect.

Also, minimal versions of approximations always exist in classes of pure–injective modules:

Proposition 1.10. *Let R be a ring and M be a module. Let $\mathcal{C}$ be a class of pure–injective modules such that $\mathcal{C}$ is closed under direct summands. Let $f \in \mathrm{Hom}_R(M, C)$ be a $\mathcal{C}$–preenvelope of M. Then there is a decomposition $C = D \oplus E$ such that $\mathrm{Im}\, f \subseteq D$ and $f : M \to D$ is left minimal. In particular, $f : M \to D$ is a $\mathcal{C}$–envelope of M.*

Proof. The proof uses the fact that the tensor product functor (as a functor from the category Mod–R to the category, $D(R)$ of all additive functors from the category of all finitely presented left R–modules to Mod–$\mathbb{Z}$) identifies pure–injective modules with injective objects in $D(R)$. The result then follows from the existence of injective envelopes in $D(R)$. For more details, see [56, Chap.7] and [58]. ∎

The following lemma is known as the Wakamatsu Lemma (see [79, §2]). It shows that under rather weak assumptions on the class $\mathcal{C}$, $\mathcal{C}$–envelopes and $\mathcal{C}$–covers are special in the sense of the following definition:

Definition 1.11. Let $\mathcal{C} \subseteq$ Mod–R. Define
$\mathcal{C}^\perp = \mathrm{Ker}\,\mathrm{Ext}^1_R(\mathcal{C}, -) = \{N \in \text{Mod–}R \mid \mathrm{Ext}^1_R(C, N) = 0 \text{ for all } C \in \mathcal{C}\}$
${}^\perp\mathcal{C} = \mathrm{Ker}\,\mathrm{Ext}^1_R(-, \mathcal{C}) = \{N \in \text{Mod–}R \mid \mathrm{Ext}^1_R(N, C) = 0 \text{ for all } C \in \mathcal{C}\}$.
For $\mathcal{C} = \{C\}$, we write for short $C^\perp$ and ${}^\perp C$ in place of $\{C\}^\perp$ and ${}^\perp\{C\}$, respectively.
Let $M \in$ Mod–R. A $\mathcal{C}$–preenvelope $f : M \to C$ of M is called *special*, provided that f is injective and $\mathrm{Coker}\, f \in {}^\perp\mathcal{C}$.

So a special $\mathcal{C}$–preenvelope may be viewed as an exact sequence $0 \to M \xrightarrow{f} C \to D \to 0$ with $C \in \mathcal{C}$ and $D \in {}^\perp\mathcal{C}$.

Dually, a $\mathcal{C}$–precover $f : C \to M$ of M is called *special*, if f is surjective and $\operatorname{Ker} f \in \mathcal{C}^{\perp}$.

If $\mathcal{C}$ is a class of modules such that each module M has a special preenvelope (special precover) then $\mathcal{C}$ is called *special preenveloping* (*special precovering*).

Lemma 1.12. *Let $M \in \text{Mod–}R$ and $\mathcal{C} \subseteq \text{Mod–}R$ be a class closed under extensions.*

(a) *Let $f : M \to C$ be a monic $\mathcal{C}$–envelope of M. Then f is special.*

(b) *Let $f : C \to M$ be a surjective $\mathcal{C}$–cover of M. Then f is special.*

Proof. (a) By assumption, there is an exact sequence

$$0 \to M \xrightarrow{f} C \xrightarrow{g} D \to 0.$$

In order to prove that $D \in {}^{\perp}\mathcal{C}$, we take an arbitrary extension

$$0 \to C' \to X \xrightarrow{h} D \to 0$$

with $C' \in \mathcal{C}$. We will prove that h splits. First consider the pullback of g and h:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & C' \arrow[r, equal] \arrow[d] & C' \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, equal] & P \arrow[r, "\beta"] \arrow[d, "\gamma"'] & X \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & M \arrow[r, "f"] & C \arrow[r, "g"] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Since $C, C' \in \mathcal{C}$, also $P \in \mathcal{C}$ by assumption. Since f is a $\mathcal{C}$–envelope of M, there is a homomorphism $\delta : C \to P$ with $\alpha = \delta f$. Then $f = \gamma\alpha = \gamma\delta f$, so $\gamma\delta$ is an automorphism of C.

Define $i : D \to X$ by $i(g(c)) = \beta\delta(\gamma\delta)^{-1}(c)$. This is well–defined, since

$$\delta(\gamma\delta)^{-1} f(m) = \delta f(m) = \alpha(m).$$

Moreover, $hig = h\beta\delta(\gamma\delta)^{-1} = g\gamma\delta(\gamma\delta)^{-1} = g$, so $hi = \mathrm{id}_D$ and h splits.

(b) dual to (a). ∎

The $\mathcal{C}$–envelope f of a module M must be monic provided that $\mathcal{I}_0 \subseteq \mathcal{C}$. This is because $M \hookrightarrow E(M)$ factors through f. Similarly, $\mathcal{P}_0 \subseteq \mathcal{C}$ implies that any $\mathcal{C}$–cover of M is surjective.

Also notice that the lemma above holds with Mod–R replaced by its subcategory mod–R.

There is an explicit duality between special preenvelopes and special precovers discovered by Salce, arising from the notion of a cotorsion pair:

Definition 1.13. Let $\mathcal{A}, \mathcal{B} \subseteq$ Mod–R. The pair $(\mathcal{A}, \mathcal{B})$ is a *cotorsion pair* if $\mathcal{A} = {}^\perp\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^\perp$.

Let $\mathcal{C}$ be a class of modules. Then $\mathcal{C} \subseteq {}^\perp(\mathcal{C}^\perp)$ as well as $\mathcal{C} \subseteq ({}^\perp\mathcal{C})^\perp$. Moreover, $\mathfrak{G}_\mathcal{C} = ({}^\perp(\mathcal{C}^\perp), \mathcal{C}^\perp)$ and $\mathfrak{C}_\mathcal{C} = ({}^\perp\mathcal{C}, ({}^\perp\mathcal{C})^\perp)$ are easily seen to be cotorsion pairs, called the cotorsion pairs *generated* and *cogenerated*, respectively, by the class $\mathcal{C}$. (In the case when $\mathcal{C}$ consists of a single module C, we will simply write ${}^\perp C$ and $C^\perp$ in place of ${}^\perp\{C\}$ and $\{C\}^\perp$.)

If $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a cotorsion pair, then the class $\mathcal{K}_\mathfrak{C} = \mathcal{A} \cap \mathcal{B}$ is the *kernel* of $\mathfrak{C}$. Note that each element K of the kernel is a *splitter*, that is, K satisfies $\mathrm{Ext}^1_R(K, K) = 0$.

For any ring R, the class of all cotorsion pairs of modules is denoted by L_{Ext}. Note that in general, L_{Ext} is a proper class (see [45]).

L_{Ext} is partially ordered by the inclusion of the first components of cotorsion pairs. The largest element of L_{Ext} is $\mathfrak{G}_{\text{Mod}-R} = (\text{Mod}–R, \mathcal{I}_0)$, while the least is $\mathfrak{C}_{\text{Mod}-R} = (\mathcal{P}_0, \text{Mod}–R)$.

Cotorsion pairs are analogs of the classical (non–hereditary) torsion pairs, where Hom (= Ext^0) is replaced by Ext^1. Similarly, we define Tor–pairs: For a class of (right resp. left) R–modules, $\mathcal{C}$, we put

$\mathcal{C}^\intercal = \mathrm{Ker}\,\mathrm{Tor}^R_1(\mathcal{C}, -) = \{N \in R\text{–Mod} \mid \mathrm{Tor}^R_1(C, N) = 0 \text{ for all } C \in \mathcal{C}\}$,

resp.

${}^\intercal\mathcal{C} = \mathrm{Ker}\,\mathrm{Tor}^R_1(-, \mathcal{C}) = \{N \in \text{Mod–}R \mid \mathrm{Tor}^R_1(N, C) = 0 \text{ for all } C \in \mathcal{C}\}$.

$(\mathcal{A}, \mathcal{B})$ is called a Tor*–pair*, if $\mathcal{A} = {}^\intercal\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^\intercal$. In this case both $\mathcal{A}$ and $\mathcal{B}$ are closed under direct limits, since Tor commutes with direct limits. (For simplicity, we will write $A^\intercal$ and ${}^\intercal B$ rather than $\{A\}^\intercal$ and ${}^\intercal\{B\}$ for $A \in$ Mod–R and $B \in R$–Mod.)

The set of all Tor–pairs is denoted by L_{Tor} (That L_{Tor} is always a set, not a proper class, is proved in Corollary 1.53 below). L_{Tor} is partially ordered by inclusion in the first components. The least element of L_{Tor} is $(\mathcal{FL}, \text{Mod–}R)$, the largest $(\text{Mod–}R, \mathcal{FL})$, where $\mathcal{FL} = \mathcal{F}_0$ denotes the class of all flat modules.

The interesting fact is that L_{Tor} can canonically be embedded into L_{Ext}. This follows from the Ext–Tor–relations of the classical homological algebra:

Lemma 1.14. *Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a* Tor*–pair. Then $\mathfrak{D} = (\mathcal{A}, \mathcal{A}^\perp)$ is a cotorsion pair. Moreover, $\mathfrak{D} = \mathfrak{C}_\mathcal{C}$, where $\mathcal{C} = \{B^c \mid B \in \mathcal{B}\} \subseteq \mathcal{PI}$.*

Proof. The statement follows from the well-known isomorphism

$$\mathrm{Ext}^1_R(A, B^c) \cong (\mathrm{Tor}^R_1(A, B))^c,$$

where B^c denotes the character module of a left R–module B. ∎

Another reason for investigating Tor–pairs is in their relation to closures of classes of modules under forming direct limits (see Theorem 2.61 below).

Let us consider further examples of cotorsion pairs:

Example 1.15.

(i) Consider the case of Lemma 1.14, when $\mathcal{A} = \mathcal{FL}$ and $\mathcal{B} = \text{Mod–}R$. Then $(\mathcal{FL}, \mathcal{EC})$ is a cotorsion pair, the so–called *Enochs cotorsion pair.* Here

$\mathcal{EC} = \mathcal{FL}^{\perp}$ is the class of all *Enochs cotorsion* modules. By Lemma 1.14, any character module and hence any pure–injective module, is Enochs cotorsion. That is, $\mathcal{PI} \subseteq \mathcal{EC}$.

(ii) Another case of interest is when $\mathcal{A} = \mathcal{TF}$, where $\mathcal{TF} = {}^{\intercal}\mathcal{S}$, and $\mathcal{S}$ is a representative set of all cyclically presented left R–modules. (Recall that a left R–module M is *cyclically presented* provided that $M \cong R/Rr$ for some $r \in R$.) The elements of $\mathcal{TF}$ are the *torsion–free* modules. (In the particular case, when R is a domain, $M \in \mathcal{TF}$, iff $mr \neq 0$ for all $m \neq 0$ and $r \neq 0$, that is, M is torsion–free in the usual sense.) By Lemma 1.14, $(\mathcal{TF}, \mathcal{RC})$ is a cotorsion pair, the so–called *Warfield cotorsion pair.* Here $\mathcal{RC} = \mathcal{TF}^{\perp}$ is the class of all *Warfield cotorsion* modules.

(iii) Let R be a domain and Q be its quotient field. The cotorsion pair generated by Q is called the *Matlis cotorsion pair.* We will have more on this in chapters 2 and 4.

(iv) A module $M \in \text{Mod}–R$ is a *Whitehead module* provided $\text{Ext}^1_R(M, R) = 0$. The class of all Whitehead modules is denoted by $\mathcal{W}_1$. The corresponding cotorsion pair (cogenerated by R) is denoted by $\mathfrak{W}_1$ and called the *Whitehead cotorsion pair.* By [35] and [75], some of the basic properties of this cotorsion pair depend on the extension of ZFC that we work in.

Clearly, $\mathcal{P}_0 \subseteq \mathcal{FL} \subseteq \mathcal{TF}$, so $\mathcal{I}_0 \subseteq \mathcal{RC} \subseteq \mathcal{EC}$ for any ring R.

We turn to approximations induced by cotorsion pairs. First we have an immediate corollary of Lemma 1.12.

Corollary 1.16. *Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. If $\mathcal{A}$ is covering, then $\mathcal{A}$ is special precovering, and if $\mathcal{B}$ is enveloping, then $\mathcal{B}$ is special preenveloping.*

The mutually dual categorical notions of a special precover and a special preenvelope are tied up by the homological tie of a cotorsion pair. In a sense, this fact is a remedy for the non–existence of a duality involving the category of all modules over a ring.

Lemma 1.17. *Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair of modules. Then the following are equivalent:*

(a) *Each module has a special $\mathcal{A}$–precover.*

(b) *Each module has a special $\mathcal{B}$–preenvelope.*

In this case, the cotorsion pair $\mathfrak{C}$ is called complete.

Proof. (a) implies (b): let $M \in \text{Mod}–R$. There is an exact sequence

$$0 \to M \to I \xrightarrow{\pi} F \to 0,$$

where I is injective. By assumption, there is a special $\mathcal{A}$–precover ρ of F

$$0 \to B \to A \xrightarrow{\rho} F \to 0.$$

Consider the pullback of π and ρ:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & M \arrow[r, equal] \arrow[d] & M \arrow[d] & \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, "\gamma"'] & I \arrow[r] \arrow[d, "\pi"'] & 0 \\
0 \arrow[r] & B \arrow[r] & A \arrow[r, "\rho"] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Since $B, I \in \mathcal{B}$, also $P \in \mathcal{B}$. So the left–hand vertical exact sequence is a special $\mathcal{B}$–preenvelope of M.

(b) implies (a): by a dual argument. ∎

Lemma 1.17 also holds true when restricted to finitely generated modules provided that injective envelopes of finitely generated modules are finitely generated (when R is an artin algebra, for example).

There is another case when cotorsion pairs tie up dual notions. First we introduce the needed setting:

For a module M, let

$$\mathfrak{P}:\ \dots \xrightarrow{f_{n+1}} P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

be a projective resolution of M. For each $i < \omega$, the module $\operatorname{Im} f_i$ is called the *i–th syzygy* of M in $\mathfrak{P}$. We denote by $\Omega^i(M)$ the class of all the i–th syzygies occurring in all projective resolutions of M.

Let $m \geq 0$. We will say that M is FP_m, provided that M has a projective resolution $\mathfrak{P}$ such that P_n is finitely generated for each $n \leq m$. Obviously, M is FP_0, iff M is finitely generated, and M is FP_1, iff M is finitely presented. We will often deal with FP_2–modules: these are the modules isomorphic to P/F, where P is finitely generated and projective, and F is a finitely presented submodule of P.

Note that if R is a right coherent ring and M is finitely presented, then M is FP_n for all $n \geq 0$. If R is right coherent and M is finitely presented, then we will consider only projective resolutions consisting of finitely presented modules – in particular, all syzygies of M will be finitely presented.

For any ring R, we denote by mod–R the class of all modules possessing a projective resolution consisting of finitely generated modules. (mod–R coincides with the class of all modules M that are FP_m for all $m < \omega$ — see [29, VIII.4]).

Given an injective coresolution of M,

$$\mathfrak{I}:\ 0 \longrightarrow M \xrightarrow{g_0} I_0 \xrightarrow{g_1} \dots \xrightarrow{g_{n-1}} I_n \xrightarrow{g_n} I_{n+1} \xrightarrow{g_{n+1}} \dots,$$

the module $\operatorname{Im} g_i$ is called the *i–th cosyzygy* of M in $\mathfrak{I}$. We will denote by $\Omega^{-i}(M)$ the class of all the i–th cosyzygies occurring in all injective coresolutions of M.

Let $\mathcal{C} \subseteq \text{Mod–}R$. For an integer i, define $\Omega^i(\mathcal{C}) = \bigcup_{M\in\mathcal{C}} \Omega^i(M)$.

We will often use the so called *dimension shifting*, that is, the computation of the Ext–groups using syzygies and/or cosyzygies of modules as follows:

$$\operatorname{Ext}^n_R(M,N) \cong \operatorname{Ext}^1_R(\Omega^{n-1}(M),N) \cong \operatorname{Ext}^1_R(M,\Omega^{-n+1}(N))$$

for all $M,N \in \text{Mod–}R$ and $n \geq 1$. Similarly,

$$\operatorname{Tor}^R_n(M,N) \cong \operatorname{Tor}^R_1(\Omega^{n-1}(M),N) \cong \operatorname{Tor}^R_1(M,\Omega^{n-1}(N))$$

for all $M \in \text{Mod–}R$, $N \in R\text{–Mod}$ and $n \geq 1$.

Definition 1.18. Let R be a ring and $\mathcal{C}$ be a class of modules.

(i) $\mathcal{C}$ is *resolving*, provided that $\mathcal{C}$ is closed under extensions, $\mathcal{P}_0 \subseteq \mathcal{C}$, and $A \in \mathcal{C}$, whenever $0 \to A \to B \to C \to 0$ is a short exact sequence such that $B, C \in \mathcal{C}$.

(ii) $\mathcal{C}$ is *coresolving*, provided that $\mathcal{C}$ is closed under extensions, $\mathcal{I}_0 \subseteq \mathcal{C}$, and $C \in \mathcal{C}$, whenever $0 \to A \to B \to C \to 0$ is a short exact sequence such that $A, B \in \mathcal{C}$.

(iii) $\mathcal{C}$ is *syzygy closed*, provided that $\Omega^1(\mathcal{C}) \subseteq \mathcal{C}$ (and hence $\Omega^i(\mathcal{C}) \subseteq \mathcal{C}$ for all $i < \omega$).

(iv) $\mathcal{C}$ is *cosyzygy closed*, provided that $\Omega^{-1}(\mathcal{C}) \subseteq \mathcal{C}$ (and hence $\Omega^{-i}(\mathcal{C}) \subseteq \mathcal{C}$ for all $i < \omega$).

(v) Let $1 \leq n < \omega$. Define
$\mathcal{C}^{\perp_n} = \operatorname{Ker}\operatorname{Ext}^n_R(\mathcal{C},-) = \{N \in \text{Mod–}R \mid \operatorname{Ext}^n_R(C,N) = 0 \ \forall\, C \in \mathcal{C}\}$,
${}^{\perp_n}\mathcal{C} = \operatorname{Ker}\operatorname{Ext}^n_R(-,\mathcal{C}) = \{N \in \text{Mod–}R \mid \operatorname{Ext}^n_R(N,C) = 0 \ \forall\, C \in \mathcal{C}\}$.
In particular, $\mathcal{C}^{\perp_1} = \mathcal{C}^{\perp}$ and ${}^{\perp_1}\mathcal{C} = {}^{\perp}\mathcal{C}$. Moreover, we define
$\mathcal{C}^{\perp_\infty} = \bigcap_{1\leq n<\omega} \mathcal{C}^{\perp_n}$,
${}^{\perp_\infty}\mathcal{C} = \bigcap_{1\leq n<\omega} {}^{\perp_n}\mathcal{C}$.

We record a couple of easy properties of the notions defined above:

Lemma 1.19. *Let R be a ring and $\mathcal{C}$ be a class of modules.*

(a) *The class ${}^{\perp_\infty}\mathcal{C}$ is resolving, and $\mathcal{C}^{\perp_\infty}$ coresolving.*

(b) *Any resolving class is syzygy closed; any coresolving class is cosyzygy closed.*

(c) *Let $i < \omega$. If $\mathcal{C}$ is syzygy closed, then so is $\Omega^i(\mathcal{C})$. If $\mathcal{C}$ is cosyzygy closed, then so is $\Omega^{-i}(\mathcal{C})$.*

(d) *Let $k < i < \omega$. Then $\mathcal{C}^{\perp_i} = (\Omega^k(\mathcal{C}))^{\perp_{i-k}}$ and ${}^{\perp_i}\mathcal{C} = {}^{\perp_{i-k}}(\Omega^{-k}(\mathcal{C}))$.*

For example, the classes $\mathcal{P}_0$ and $\mathcal{FL}$ are always resolving. $\mathcal{TF}$ is resolving, provided that $\operatorname{Tor}^R_1(M,Rr) = 0$ for each $r \in R$ and each torsion–free module M. In particular, $\mathcal{TF}$ is resolving when R is a commutative domain.

Lemma 1.20. *Let R be a ring and $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then the following assertions are equivalent:*

(a) *$\mathcal{A}$ is resolving;*

(b) *$\mathcal{B}$ is coresolving;*

(c) $\operatorname{Ext}^i_R(A, B) = 0$ *for all $i \geq 1$, $A \in \mathcal{A}$ and $B \in \mathcal{B}$.*

In this case, the cotorsion pair $\mathfrak{C}$ is called hereditary.

Proof. (a) implies (c) and (b): let $0 \to C \to P \to A \to 0$ be an exact sequence with $A \in \mathcal{A}$ and $P \in \mathcal{P}_0$. By the premise, $C \in \mathcal{A}$. Let $B \in \mathcal{B}$. Applying $\operatorname{Hom}_R(-, B)$, we get the exact sequence $0 = \operatorname{Ext}^1_R(C, B) \to \operatorname{Ext}^2_R(A, B) \to \operatorname{Ext}^2_R(P, B) = 0$. By induction, we get (c).

In order to prove (b), we take an exact sequence $0 \to E \to F \to G \to 0$ with $E, F \in \mathcal{B}$. Consider $A \in \mathcal{A}$. Applying $\operatorname{Hom}_R(A, -)$, we get the exact sequence $0 = \operatorname{Ext}^1_R(A, F) \to \operatorname{Ext}^1_R(A, G) \to \operatorname{Ext}^2_R(A, E) = 0$. This proves that $G \in \mathcal{A}^\perp = \mathcal{B}$.

(b) implies (c): by a dual argument.

(c) implies (a): let $0 \to E \to D \to C \to 0$ be an exact sequence of modules such that $C, D \in \mathcal{A}$. Take $B \in \mathcal{B}$ and apply $\operatorname{Hom}_R(-, B)$. Then the sequence $0 = \operatorname{Ext}^1_R(D, B) \to \operatorname{Ext}^1_R(E, B) \to \operatorname{Ext}^2_R(C, B) = 0$ is exact, whence $E \in \mathcal{A}$. ∎

An easy consequence of the two lemmas above says that in certain cases we need not distinguish between $\perp$ and $\perp_\infty$:

Corollary 1.21. *Let R be a ring and $\mathcal{C}$ be a class of modules.*

(a) *Assume that $\mathcal{C}$ is syzygy closed. Then $\mathcal{C}^\perp = \mathcal{C}^{\perp_\infty}$ is coresolving and ${}^\perp(\mathcal{C}^\perp) = {}^{\perp_\infty}(\mathcal{C}^{\perp_\infty})$ is resolving. The cotorsion pair generated by $\mathcal{C}$ is hereditary.*

(b) *Assume that $\mathcal{C}$ is cosyzygy closed. Then ${}^\perp\mathcal{C} = {}^{\perp_\infty}\mathcal{C}$ is resolving and $({}^\perp\mathcal{C})^\perp = ({}^{\perp_\infty}\mathcal{C})^{\perp_\infty}$ is coresolving. The cotorsion pair cogenerated by $\mathcal{C}$ is hereditary.*

In the sequel, we will prove that almost all cotorsion pairs are complete, so they provide for approximations. In some cases minimal approximations exist, that is, the cotorsion pairs are perfect in the sense of the following definition:

Definition 1.22. Let R be a ring, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair.

(i) $\mathfrak{C}$ is called *perfect*, provided that $\mathcal{A}$ is a covering class and $\mathcal{B}$ is an enveloping class.

(ii) $\mathfrak{C}$ is called *closed*, provided that $\mathcal{A} = \varinjlim \mathcal{A}$, that is, the class $\mathcal{A}$ is closed under forming direct limits.

The term "perfect" comes from the classical result of Bass characterizing right perfect rings by the property that the cotorsion pair $\mathfrak{P}_0 = (\mathcal{P}_0, \text{Mod–}R)$ is perfect (cf. [2, §28]).

Clearly, any perfect cotorsion pair is complete. The converse fails in general: for example, $\mathfrak{P}_0$ is complete for any ring. Numerous examples of perfect and/or complete cotorsion pairs will appear in the sequel.

In order to prove the existence of minimal approximations, we will often use the following version of a result due to Enochs and Xu [79, §2.2]:

Theorem 1.23. *Let R be a ring and M be a module. Let $\mathcal{C}$ be a class of modules closed under extensions and direct limits. Assume that M has a special $\mathcal{C}^\perp$–preenvelope ν with $\operatorname{Coker}\nu \in \mathcal{C}$. Then M has a $\mathcal{C}^\perp$–envelope.*

Proof. By an ad hoc notation, we will call an exact sequence $0 \to M \to F \to C \to 0$ with $C \in \mathcal{C}$ an *Ext–generator*, provided that for each exact sequence $0 \to M \to F' \to C' \to 0$ with $C' \in \mathcal{C}$ there exist $f \in \operatorname{Hom}_R(F', F)$ and $g \in \operatorname{Hom}_R(C', C)$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F' \arrow[r] \arrow[d, "f"] & C' \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & M \arrow[r] & F \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

is commutative. By assumption, there exists an Ext–generator with the middle term $F \in \mathcal{C}^\perp$. The proof is divided into three steps:

Lemma 1.24. *Assume $0 \to M \to F \to C \to 0$ is an Ext–generator. Then there exist an Ext–generator $0 \to M \to F' \to C' \to 0$ and a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & M \arrow[r] & F' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

such that $\operatorname{Ker}(f) = \operatorname{Ker}(f'f)$ holds in any commutative diagram whose rows are Ext–generators:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F' \arrow[r] \arrow[d, "f'"] & C' \arrow[r] \arrow[d, "g'"] & 0 \\
0 \arrow[r] & M \arrow[r] & F'' \arrow[r] & C'' \arrow[r] & 0.
\end{tikzcd}$$

Proof. Assume that the assertion does not hold. By induction, we will construct a direct system of Ext–generators indexed by ordinals as follows: First let the second row be the same as the first one, that is, put $F' = F_0 = F$, $C' = C_0 = C$, $f = \mathrm{id}_F$ and $g = \mathrm{id}_C$. Then there exist $F_1 = F''$, $C_1 = C''$, $f_{10} = f'$ and $g_{10} = g'$ such that the diagram above commutes, its rows are Ext–generators and $\operatorname{Ker} f_{10} \supsetneq \operatorname{Ker} f = 0$.

Assume that the Ext–generator $0 \to M \to F_\alpha \to C_\alpha \to 0$ is defined together with $f_{\alpha\beta} \in \operatorname{Hom}_R(F_\beta, F_\alpha)$ and $g_{\alpha\beta} \in \operatorname{Hom}_R(C_\beta, C_\alpha)$ for all $\beta \leq \alpha$. Then there exist $F_{\alpha+1}$, $C_{\alpha+1} \in \mathcal{C}$, $f_{\alpha+1,\alpha}$ and $g_{\alpha+1,\alpha}$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F_\alpha & \longrightarrow & C_\alpha & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f_{\alpha+1,\alpha}} & & \downarrow{\scriptstyle g_{\alpha+1,\alpha}} & & \\ 0 & \longrightarrow & M & \longrightarrow & F_{\alpha+1} & \longrightarrow & C_{\alpha+1} & \longrightarrow & 0 \end{array}$$

commutes, its rows are Ext–generators and $\operatorname{Ker} f_{\alpha+1,0} \supsetneq \operatorname{Ker} f_{\alpha 0}$, where $f_{\alpha+1,\beta} = f_{\alpha+1,\alpha} f_{\alpha\beta}$ and $g_{\alpha+1,\beta} = g_{\alpha+1,\alpha} g_{\alpha\beta}$ for all $\beta \leq \alpha$.

If α is a limit ordinal, put $F_\alpha = \varinjlim_{\beta<\alpha} F_\beta$ and $C_\alpha = \varinjlim_{\beta<\alpha} C_\beta$. Consider the direct limit $0 \to M \to F_\alpha \to C_\alpha \to 0$ of the Ext–generators $0 \to M \to F_\beta \to C_\beta \to 0$, $(\beta < \alpha)$. Since $\mathcal{C}$ is closed under direct limits, we have $C_\alpha \in \mathcal{C}$. Since $0 \to M \to F_\beta \to C_\beta \to 0$ is an Ext–generator for (some) $\beta < \alpha$, also $0 \to M \to F_\alpha \to C_\alpha \to 0$ is an Ext–generator.

Put $f_{\alpha\beta} = \varinjlim_{\beta \leq \beta' < \alpha} f_{\beta'\beta}$ and $g_{\alpha\beta} = \varinjlim_{\beta \leq \beta' < \alpha} g_{\beta'\beta}$ for all $\beta < \alpha$. Then $\operatorname{Ker}(f_{\alpha 0}) \supseteq \operatorname{Ker}(f_{\beta 0})$, and hence $\operatorname{Ker}(f_{\alpha 0}) \supsetneq \operatorname{Ker}(f_{\beta 0})$, for each $\beta < \alpha$.

By induction, for each α we obtain a strictly increasing chain $(\operatorname{Ker} f_{\beta 0} \mid \beta < \alpha)$, consisting of submodules of F, a contradiction. ■

Lemma 1.25. *Assume $0 \to M \to F \to C \to 0$ is an Ext–generator. Then there exist an Ext–generator $0 \to M \to F' \to C' \to 0$ and a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

such that $\operatorname{Ker}(f') = 0$ *in any commutative diagram whose rows are Ext–generators:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} & & \\ 0 & \longrightarrow & M & \longrightarrow & F'' & \longrightarrow & C'' & \longrightarrow & 0. \end{array}$$

Proof. By induction on $n < \omega$, we infer from Lemma 1.24 that there is a countable direct system $\mathcal{D}$ of Ext–generators $0 \to M \to F_n \to C_n \to 0$ with homomorphisms $f_{n+1,n} \in \operatorname{Hom}_R(F_n, F_{n+1})$, $g_{n+1,n} \in \operatorname{Hom}_R(C_n, C_{n+1})$, such that the 0–th term of $\mathcal{D}$ is the given Ext–generator $0 \to M \to F \to C \to 0$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F_n & \longrightarrow & C_n & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f_{n+1,n}} & & \downarrow{\scriptstyle g_{n+1,n}} & & \\ 0 & \longrightarrow & M & \longrightarrow & F_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & 0 \end{array}$$

is commutative, and for each commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle \bar{g}} & & \\ 0 & \longrightarrow & M & \longrightarrow & \bar{F} & \longrightarrow & \bar{C} & \longrightarrow & 0 \end{array}$$

whose rows are Ext–generators, we have $\operatorname{Ker}(f_{n+1,n}) = \operatorname{Ker}(\bar{f} f_{n+1,n})$.

Consider the direct limit $0 \to M \to F' \to C' \to 0$ of $\mathcal{D}$, so $F' = \varinjlim_{n<\omega} F_n$ and $C' = \varinjlim_{n<\omega} C_n$. Since $\mathcal{C}$ is closed under direct limits, we have $C' \in \mathcal{C}$, and $0 \to M \to F' \to C' \to 0$ is an Ext–generator. It is easy to check that this generator has the required injectivity property. ■

Lemma 1.26. *Let $0 \to M \xrightarrow{\nu} F' \xrightarrow{\pi} C' \to 0$ be the Ext–generator constructed in Lemma 1.25. Then $\nu : M \to F'$ is a $\mathcal{C}^\perp$–envelope of M.*

Proof. First we prove that in each commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F' \arrow[r] \arrow[d, "f'"'] & C' \arrow[r] \arrow[d, "g'"'] & 0 \\
0 \arrow[r] & M \arrow[r] & F' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

f' is an automorphism.

Assume this is not true. By induction, we construct a direct system of Ext–generators, $0 \to M \to F_\alpha \to C_\alpha \to 0$, indexed by ordinals, with injective, but not surjective, homomorphisms $f_{\alpha\beta} \in \operatorname{Hom}_R(F_\beta, F_\alpha)$ $(\beta < \alpha)$. In view of Lemma 1.25, we take

$$0 \to M \to F_\alpha \to C_\alpha \to 0 = 0 \to M \xrightarrow{\nu} F' \xrightarrow{\pi} C' \to 0$$

in case $\alpha = 0$ or α non–limit, and $F_\alpha = \varinjlim F_\beta$, and $C_\alpha = \varinjlim C_\beta$ if α is a limit ordinal. Then for each non–limit ordinal α, $(\operatorname{Im} f_{\alpha\beta} \mid \beta \text{ non–limit}, \beta < \alpha)$ is a strictly increasing sequence of submodules of F', a contradiction.

It remains to prove that $F' \in \mathcal{C}^\perp$. Consider an exact sequence $0 \to F' \xrightarrow{\mu} X \to C \to 0$, where $C \in \mathcal{C}$. We will prove that this sequence splits.
Consider the pushout of π and μ:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "\nu"] \arrow[d, equal] & F' \arrow[r, "\pi"] \arrow[d, "\mu"'] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & X \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
 & & C \arrow[r, equal] \arrow[d] & C \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

Since $\mathcal{C}$ is closed under extensions, we have $P \in \mathcal{C}$. Since $0 \to M \xrightarrow{\nu} F' \xrightarrow{\pi} C' \to 0$ is an Ext–generator, we also have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\mu'"'] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "\nu"] & F' \arrow[r, "\pi"] & C' \arrow[r] & 0.
\end{tikzcd}$$

By the first part of the proof, $\mu'\mu$ is an automorphism of F'. It follows that $0 \to F' \xrightarrow{\mu} X \to C \to 0$ splits. ∎

Theorem 1.27. *Let R be a ring, M be a module, and $\mathcal{C}$ be a class of modules closed under direct limits. Assume that M has a $\mathcal{C}$–precover. Then M has a $\mathcal{C}$–cover.*

Proof. The proof is by a construction of precovers with additional injectivity properties. The three steps are analogous to Lemmas 1.24 - 1.26 (see [79, §2.2]). ∎

Corollary 1.28. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete and closed cotorsion pair. Then $\mathfrak{C}$ is perfect.*

Proof. By Theorems 1.23 and 1.27. ∎

Our next goal is to show that complete cotorsion pairs are abundant. We will prove that any cotorsion pair generated by a set of modules is complete, and any cotorsion pair cogenerated by a class of pure–injective modules is perfect.

Definition 1.29.

(i) Let μ be an ordinal and $\mathcal{A} = (A_\alpha \mid \alpha \le \mu)$ be a (well–ordered) sequence of modules. Then $\mathcal{A}$ is called a *continuous chain of modules* provided that $A_0 = 0$, $A_\alpha \subseteq A_{\alpha+1}$ for all $\alpha < \mu$, and $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for all limit ordinals $\alpha \le \mu$.

(ii) Let M be a module and $\mathcal{C}$ be a class of modules. M is $\mathcal{C}$*–filtered* provided that there are an ordinal κ and a continuous chain of submodules of M, $(M_\alpha \mid \alpha \le \kappa)$, such that $M = M_\kappa$, and each of the modules $M_{\alpha+1}/M_\alpha$ ($\alpha < \kappa$) is isomorphic to an element of $\mathcal{C}$. The chain $(M_\alpha \mid \alpha \le \kappa)$ is called a $\mathcal{C}$*–filtration* of M. If κ is finite, then M is said to be *finitely* $\mathcal{C}$*–filtered.*

(iii) Similarly, we define *continuous direct systems of exact sequences* for well–ordered direct systems of short exact sequences of modules.

For example, if $\mathcal{C} = \operatorname{simp} R$, then the $\mathcal{C}$–filtered modules coincide with the *semiartinian modules*, while the finitely $\mathcal{C}$–filtered modules are exactly the *modules of finite length.*

The following lemma gives an important sufficient condition for the vanishing of Ext.

Lemma 1.30. *Let N be a module, and M be a ${}^\perp N$–filtered module. Then $M \in {}^\perp N$.*

Proof. Let $(M_\alpha \mid \alpha \le \kappa)$ be a ${}^\perp N$–filtration of M. So $\operatorname{Ext}^1_R(M_0, N) = 0$ and, for each $\alpha < \kappa$, $\operatorname{Ext}^1_R(M_{\alpha+1}/M_\alpha, N) = 0$. We will prove $\operatorname{Ext}^1_R(M, N) = 0$.

By induction on $\alpha \le \kappa$ we will prove that $\operatorname{Ext}^1_R(M_\alpha, N) = 0$. This is clear for $\alpha = 0$.

The exact sequence

$$0 = \operatorname{Ext}^1_R(M_{\alpha+1}/M_\alpha, N) \to \operatorname{Ext}^1_R(M_{\alpha+1}, N) \to \operatorname{Ext}^1_R(M_\alpha, N) = 0$$

proves the induction step.

Assume $\alpha \leq \kappa$ is a limit ordinal. Let $0 \to N \to I \xrightarrow{\pi} I/N \to 0$ be an exact sequence with I an injective module. In order to prove that $\mathrm{Ext}^1_R(M_\alpha, N) = 0$, we show that the abelian group homomorphism $\mathrm{Hom}_R(M_\alpha, \pi) : \mathrm{Hom}_R(M_\alpha, I) \to \mathrm{Hom}_R(M_\alpha, I/N)$ is surjective.

Let $\varphi \in \mathrm{Hom}_R(M_\alpha, I/N)$. By induction we define $\psi_\beta \in \mathrm{Hom}_R(M_\beta, I)$, $\beta < \alpha$, so that $\varphi \restriction M_\beta = \pi\psi_\beta$ and $\psi_\beta \restriction M_\gamma = \psi_\gamma$ for all $\gamma < \beta < \alpha$.

First define $M_{-1} = 0$ and $\psi_{-1} = 0$. If ψ_β is already defined, the injectivity of I yields the existence of $\eta \in \mathrm{Hom}_R(M_{\beta+1}, I)$ such that $\eta \restriction M_\beta = \psi_\beta$. Put $\delta = \varphi \restriction M_{\beta+1} - \pi\eta \in \mathrm{Hom}_R(M_{\beta+1}, I/N)$. Then $\delta \restriction M_\beta = 0$. Since $\mathrm{Ext}^1_R(M_{\beta+1}/M_\beta, N) = 0$, there is $\epsilon \in \mathrm{Hom}_R(M_{\beta+1}, I)$ such that $\epsilon \restriction M_\beta = 0$ and $\pi\epsilon = \delta$. Put $\psi_{\beta+1} = \eta + \epsilon$. Then $\psi_{\beta+1} \restriction M_\beta = \psi_\beta$ and $\pi\psi_{\beta+1} = \pi\eta + \delta = \varphi \restriction M_{\beta+1}$. For a limit ordinal $\beta < \alpha$, put $\psi_\beta = \bigcup_{\gamma<\beta} \psi_\gamma$.

Finally, put $\psi_\alpha = \bigcup_{\beta<\alpha} \psi_\beta$. By the construction, $\pi\psi_\alpha = \varphi$.
The claim is just the case of $\alpha = \kappa$. ■

There is a version of Lemma 1.30 for Tor. Before proving it, we recall the well–known relations between the functors Ext and Tor:

Lemma 1.31. *Let R and S be rings and let A be a module.*

(a) *Let $B \in R$–Mod–S and $C \in$ Mod–S. Then there is a natural (= functorial in each variable) isomorphism*

$$\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \cong \mathrm{Hom}_S(A \otimes_R B, C).$$

(b) *Let $n < \omega$, $B \in R$–Mod–S, and let C be an injective right S–module. Then*

$$\mathrm{Ext}^n_R(A, \mathrm{Hom}_S(B, C)) \cong \mathrm{Hom}_S(\mathrm{Tor}^R_n(A, B), C).$$

(c) *Assume A is finitely presented. Let $B \in S$–Mod–R and C an injective left S–module. Then there is a natural isomorphism*

$$A \otimes_R \mathrm{Hom}_S(B, C) \cong \mathrm{Hom}_S(\mathrm{Hom}_R(A, B), C).$$

(d) *Let $m < \omega$. Assume A has a projective resolution $\ldots \to P_n \to \ldots \to P_0 \to A \to 0$ such that P_i is finitely generated for each $i \leq m+1$. Moreover, let $B \in S$–Mod–R and C an injective left S–module. Then*

$$\mathrm{Tor}^R_i(A, \mathrm{Hom}_S(B, C)) \cong \mathrm{Hom}_S(\mathrm{Ext}^i_R(A, B), C)$$

for each $i \leq m$.

Corollary 1.32. *Let N be a left R–module, and M be a ${}^\intercal N$–filtered module. Then $M \in {}^\intercal N$.*

Proof. By Lemmas 1.31 (b) and 1.30. ■

Another immediate consequence of Lemma 1.30 is

Lemma 1.33. *Let $n < \omega$ and let M be a module. Assume M is $\mathcal{P}_n$–filtered. Then $M \in \mathcal{P}_n$.*

Proof. By dimension shifting we have $M \in \mathcal{P}_n$, iff $M \in {}^\perp\mathcal{S}_n$, where $\mathcal{S}_n = \{\Omega^n(N) \mid N \in \text{Mod–}R\}$. The claim now follows from Lemma 1.30. ∎

If M is finitely presented, then the covariant functor $\operatorname{Hom}_R(M,-)$ commutes with direct limits. This extends to the covariant Ext functor as follows:

Lemma 1.34. *Let R be a ring, $n \geq 0$ and M be an FP_{n+1}–module (for example, let R be right coherent and M finitely presented). Let $\{N_\alpha, f_{\beta\alpha} \mid \alpha \leq \beta \in I\}$ be a direct system of modules. Then for all $i \leq n$*

$$\operatorname{Ext}^i_R(M, \varinjlim_{\alpha\in I} N_\alpha) \cong \varinjlim_{\alpha\in I} \operatorname{Ext}^i_R(M, N_\alpha)\,.$$

Proof. Well–known (see e.g. [40, §10]) ∎

There is a dual result for the contravariant Ext functors (for its proof, we refer to [46]):

Lemma 1.35. *Let R be a ring and M be a pure–injective module. Let $\{N_\alpha, f_{\beta\alpha} \mid \alpha \leq \beta \in I\}$ be a direct system of modules. Then for each $n \geq 0$,*

$$\operatorname{Ext}^n_R(\varinjlim_{\alpha\in I} N_\alpha, M) \cong \varprojlim_{\alpha\in I} \operatorname{Ext}^n_R(N_\alpha, M)\,.$$

Lemma 1.34 has an immediate corollary:

Corollary 1.36. *Let R be a right noetherian ring and $m < \omega$. Then the class $\mathcal{I}_m$ is closed under direct limits.*

Proof. First let $m = 0$. By assumption, any cyclic module is finitely presented, so the Baer Criterion and Lemma 1.34 for $n = 1$ show that $\mathcal{I}_0$ is closed under direct limits. For right noetherian rings, all syzygies of a finitely presented module can be taken finitely presented. So the result for $n > 0$ follows by dimension shifting. ∎

Now we introduce the notion of a definable class of modules:

Definition 1.37. Let $\mathcal{C}$ be a class of modules. Then $\mathcal{C}$ is *definable*, provided that $\mathcal{C}$ is closed under direct limits, direct products and pure submodules.

The term definable comes from the fact that the modules in $\mathcal{C}$ are axiomatized by particular formulas of the first order language of the theory of modules (see e.g. [56]).

Any definable class $\mathcal{C}$ is completely determined by its pure–injective elements:

Lemma 1.38. *Let R be a ring, M a module and $\mathcal{C}$ a definable class of modules. Then $M \in \mathcal{C}$, iff $PE(M) \in \mathcal{C}$.*

Proof. Assume $M \in \mathcal{C}$. Since M is elementarily equivalent to $PE(M)$ (see [56]), $PE(M)$ is a direct summand in an ultrapower U of M. However, any ultrapower is isomorphic to a direct limit of direct products of copies of M. So $U \in \mathcal{C}$, and hence $PE(M) \in \mathcal{C}$.

Since M is a pure submodule in $PE(M)$, the reverse implication is clear. ∎

By a result of Ziegler, any pure–injective module is elementarily equivalent to a pure–injective hull of a direct sum of indecomposable pure–injective modules. So definable classes are completely characterized by their indecomposable pure–injective elements. For more on the model–theoretic and functor–categorical approach to definable classes we refer to [34] and [56].

We will often work with the following example of a definable class:

Example 1.39. Let R be a ring and $\mathcal{C}$ be a class of FP_2–modules. Then the class $\mathcal{C}^\perp$ is definable. Indeed, since $\mathcal{C}$ consists of finitely presented modules, $\mathcal{C}^\perp$ is closed under pure submodules (and, obviously, under direct products). By Lemma 1.34, $\mathcal{C}^\perp$ is closed under direct limits.

The following theorem shows that complete cotorsion pairs are abundant:

Theorem 1.40. *Let $\mathcal{S}$ be a set of modules.*

(a) *Let M be a module. Then there is a short exact sequence*

$$0 \to M \hookrightarrow P \to N \to 0,$$

where $P \in \mathcal{S}^\perp$ and N is $\mathcal{S}$–filtered. In particular, $M \hookrightarrow P$ is a special $\mathcal{S}^\perp$–preenvelope of M.

(b) *The cotorsion pair $({}^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp)$ is complete.*

Proof. (a) Put $X = \bigoplus_{S \in \mathcal{S}} S$. Then $X^\perp = \mathcal{S}^\perp$. So w.l.o.g., we assume that $\mathcal{S}$ consists of a single module S.

Let $0 \to K \xrightarrow{\mu} F \to S \to 0$ be a short exact sequence with F a free module. Let λ be an infinite regular cardinal such that K is $< \lambda$–generated.

By induction we define an increasing chain $(P_\alpha \mid \alpha < \lambda)$ as follows:
First let $P_0 = M$. For $\alpha < \lambda$, choose the index set $I_\alpha = \mathrm{Hom}_R(K, P_\alpha)$. We define μ_α as the direct sum of I_α copies of the homomorphism μ, i.e.

$$\mu_\alpha = \mu^{(I_\alpha)} \in \mathrm{Hom}_R(K^{(I_\alpha)}, F^{(I_\alpha)}).$$

Then μ_α is a monomorphism, and $\mathrm{Coker}\, \mu_\alpha$ is isomorphic to a direct sum of copies of S. Let $\varphi_\alpha \in \mathrm{Hom}_R(K^{(I_\alpha)}, P_\alpha)$ be the canonical morphism. Note that for each $\eta \in I_\alpha$ there exist canonical embeddings $\nu_\eta \in \mathrm{Hom}_R(K, K^{(I_\alpha)})$ and $\nu'_\eta \in \mathrm{Hom}_R(F, F^{(I_\alpha)})$ such that $\eta = \varphi_\alpha \nu_\eta$ and $\nu'_\eta \mu = \mu_\alpha \nu_\eta$.

Now $P_{\alpha+1}$ is defined via the pushout of μ_α and φ_α:

$$\begin{array}{ccc} K^{(I_\alpha)} & \xrightarrow{\mu_\alpha} & F^{(I_\alpha)} \\ {\scriptstyle \varphi_\alpha} \downarrow & & \downarrow {\scriptstyle \psi_\alpha} \\ P_\alpha & \xrightarrow{\subseteq} & P_{\alpha+1}. \end{array}$$

If $\alpha \leq \lambda$ is a limit ordinal, we put $P_\alpha = \bigcup_{\beta < \alpha} P_\beta$, so the chain is continuous. Put $P = \bigcup_{\alpha < \lambda} P_\alpha$.

We will prove that $\nu : M \hookrightarrow P$ is a special $S^\perp$–preenvelope of M.
First we prove that $P \in S^\perp$. Since F is projective, we are left to show that any $\varphi \in \mathrm{Hom}_R(K, P)$ factors through μ:

Since K is $< \lambda$–generated, there are an index $\alpha < \lambda$ and $\eta \in I_\alpha$ such that $\varphi(k) = \eta(k)$ for all $k \in K$. The pushout square gives $\psi_\alpha \mu_\alpha = \sigma_\alpha \varphi_\alpha$, where σ_α denotes the inclusion of P_α into $P_{\alpha+1}$. Altogether we have $\psi_\alpha \nu'_\eta \mu = \psi_\alpha \mu_\alpha \nu_\eta = \sigma_\alpha \varphi_\alpha \nu_\eta = \sigma_\alpha \eta$. It follows that $\varphi = \psi' \mu$, where $\psi' \in \mathrm{Hom}_R(F, P)$ is defined by $\psi'(f) = \psi_\alpha \nu'_\eta(f)$ for all $f \in F$. This proves that $P \in S^\perp$.

It remains to prove that $N = P/M \in {}^\perp(\mathcal{S}^\perp)$. By construction, N is the union of the continuous chain $(N_\alpha \mid \alpha < \lambda)$, where $N_\alpha = P_\alpha/M$.

Since $P_{\alpha+1}/P_\alpha$ is isomorphic to a direct sum of copies of S by the pushout construction, so is $N_{\alpha+1}/N_\alpha \cong P_{\alpha+1}/P_\alpha$. Since $S \in {}^\perp(\mathcal{S}^\perp)$, Lemma 1.30 shows that $N \in {}^\perp(\mathcal{S}^\perp)$.

(b) follows by part (a) (cf. Lemma 1.17). ∎

Any cotorsion pair generated by a set of modules $\mathcal{S}$ is also generated by the single module $M = \bigoplus_{S \in \mathcal{S}} S$. So the following corollary of Theorem 1.40 provides a characterization of the (complete) cotorsion pairs generated by sets of modules:

Corollary 1.41. *Let M be a module. Denote by $\mathcal{Z}_M$ the class of all modules Z such that there is an exact sequence $0 \to F \to Z \to G \to 0$, where F is free and G is $\{M\}$–filtered. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. The following are equivalent*

(a) *$\mathfrak{C}$ is generated by M (that is, $\mathcal{B} = M^\perp$).*

(b) *$\mathcal{A}$ consists of all direct summands of elements of $\mathcal{Z}_M$ (and for each $A \in \mathcal{A}$, there are $Z \in \mathcal{Z}_M$ and $C \in \mathcal{K}_{\mathfrak{C}}$ such that $A \oplus C \cong Z$).*

Proof. (a) implies (b): by assumption, $\mathcal{B} = M^\perp$. Take $A \in \mathcal{A}$ and let $0 \to N \xrightarrow{\mu} F \to A \to 0$ be a short exact sequence with F free. By Theorem 1.40 (a), there is a special $\mathcal{B}$–preenvelope, $\nu : N \hookrightarrow C$ of N such that $G = C/N$ is $\{M\}$–filtered. Let $(G_\alpha \mid \alpha \leq \lambda)$ be an $\{M\}$–filtration of G. Consider the pushout of μ and ν:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & N \arrow[r, "\mu"] \arrow[d, "\nu"'] & F \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "\pi"'] & A \arrow[r] & 0 \\
 & G \arrow[r, equal] \arrow[d] & G \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

The second column gives $Z \in \mathcal{Z}_M$. The second row splits since $C \in \mathcal{B}$ and $A \in \mathcal{A}$, so $A \oplus C \cong Z$. Finally, since $F, G \in \mathcal{A}$, we have $Z \in \mathcal{A}$, so $C \in \mathcal{K}_{\mathfrak{C}}$.

(b) implies (a): by Lemma 1.30, $M^\perp = \mathcal{A}^\perp = \mathcal{B}$. ∎

Corollary 1.42. *Let $\mathcal{S}$ be a set of modules containing R. Then the class ${}^\perp(\mathcal{S}^\perp)$ consists of all direct summands of $\mathcal{S}$–filtered modules.*

Proof. By Corollary 1.41 and Lemma 1.30. ∎

In general, we cannot omit the term "direct summands" in Corollary 1.42. For example, if $\mathcal{S} = \{R\}$, then ${}^\perp(\mathcal{S}^\perp) = \mathcal{P}_0$ is the class of all projective modules while $\mathcal{S}$–filtered modules coincide with the free modules. There is, however, a way of getting rid of direct summands on the account of enlarging the set $\mathcal{S}$ (see Theorem 2.20 below).

We will see that many cotorsion pairs satisfy the equivalent conditions of Corollary 1.41. Nevertheless, this is not always the case: it is consistent with ZFC + GCH that there exist cotorsion pairs not generated by any set of modules (see [37] and [75]).

Assume that S is a module satisfying $\mathrm{Ext}^1_R(S, S^{(\kappa)}) = 0$ for all cardinals κ. Then all $\{S\}$–filtered modules are isomorphic to direct sums of copies of S. So the short exact sequence induced by the special $S^\perp$–preenvelope from Theorem 1.40 (a) takes the form $0 \to M \xrightarrow{\subseteq} P \to S^{(\lambda)} \to 0$ for a cardinal λ. This can be proved more directly, using an idea of Bongartz [28]:

Lemma 1.43. *Let R and S be rings, A be a right R–module and B be an S–R–bimodule. Let λ be the minimal number of generators of the right S–module $\mathrm{Ext}^1_R(B, A)$. Assume that $\mathrm{Ext}^1_R(B, B^{(\lambda)}) = 0$. Then there exists a module C satisfying*

(a) $\mathrm{Ext}^1_R(B, C) = 0$, *and*

(b) *there is an exact sequence* $0 \to A \xrightarrow{\mu} C \to B^{(\lambda)} \to 0$ *in* Mod–R.

In particular, μ is a special $B^\perp$–preenvelope of A.

Proof. Consider a set of extensions

$$\mathcal{E}_\alpha \,:\, 0 \to A \to E_\alpha \to B \to 0$$

such that the equivalence classes of all $\mathcal{E}_\alpha$ ($\alpha < \lambda$) generate $\mathrm{Ext}^1_R(B, A)$ as a right S–module. Let

$$\mathcal{E} \,:\, 0 \to A \to C \xrightarrow{\pi} B^{(\lambda)} \to 0$$

be the extension obtained by pushing out the direct sum extension

$$\mathcal{D} \,:\, 0 \to A^{(\lambda)} \to \oplus \sum_{\alpha<\lambda} E_\alpha \to B^{(\lambda)} \to 0$$

along the summation map $\Sigma_A : A^{(\lambda)} \to A$ defined by $\Sigma_A((a_\alpha \mid \alpha < \lambda)) = \sum_{\alpha<\lambda} a_\alpha$.

Consider the long exact sequence

$$\dots \to \mathrm{Hom}_R(B, A) \to \mathrm{Hom}_R(B, C) \to \mathrm{Hom}_R(B, B^{(\lambda)}) \xrightarrow{\delta} \mathrm{Ext}^1_R(B, A) \to$$

$$\to \mathrm{Ext}^1_R(B, C) \xrightarrow{\mathrm{Ext}^1_R(B,\pi)} \mathrm{Ext}^1_R(B, B^{(\lambda)}) = 0 \to \dots$$

induced by $\mathcal{E}$. Since equivalence classes of the extensions $\mathcal{E}_\alpha$ generate the right S–module $\operatorname{Ext}^1_R(B, A)$, the connecting S–homomorphism δ is surjective. So the S–homomorphism $\operatorname{Ext}^1_R(B, \pi)$ is monic, and $\operatorname{Ext}^1_R(B, C) = 0$. ∎

Next we prove that cotorsion pairs cogenerated by classes of pure–injective modules are complete and closed, hence they are perfect. We will follow the approach of [39], that is, we will prove the result by an application of Theorem 1.40.

Definition 1.44. For any module A and any cardinal κ, a *κ–refinement of A* (of length σ) is a continuous chain of modules, $(A_\alpha \mid \alpha \leq \sigma)$, consisting of pure submodules of A such that $A_\sigma = A$ and $|A_{\alpha+1}/A_\alpha| \leq \kappa$ for all $\alpha < \sigma$.

Now, we recall without proof several easy and well–known properties of pure embeddings:

Lemma 1.45. *Let $\lambda \geq |R| + \aleph_0$.*

(a) *Let M be a module and X a subset of M with $|X| \leq \lambda$. Then there is a pure submodule $N \subseteq_* M$ such that $X \subseteq N$ and $|N| \leq \lambda$.*

(b) *Assume $C \subseteq B \subseteq A$, $C \subseteq_* A$ and $B/C \subseteq_* A/C$. Then $B \subseteq_* A$.*

(c) *If $A \subseteq_* B$ and $B \subseteq_* C$, then $A \subseteq_* C$.*

(d) *Assume $A_0 \subseteq \dots \subseteq A_\alpha \subseteq A_{\alpha+1} \subseteq \dots$ is a chain of pure submodules of M. Then $\bigcup_\alpha A_\alpha$ is a pure submodule of M.*

The next lemma shows the role of the κ–refinements:

Lemma 1.46. *Let $\kappa = |R| + \aleph_0$. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair cogenerated by a class $\mathcal{C} \subseteq \mathcal{PI}$. Then the following are equivalent.*

(a) $A \in \mathcal{A}$.

(b) *There is a cardinal λ such that A has a κ–refinement $(A_\alpha \mid \alpha \leq \lambda)$ with $A_{\alpha+1}/A_\alpha \in \mathcal{A}$ for all $\alpha < \lambda$.*

Proof. (a) implies (b): if $|A| \leq \kappa$, we let $\lambda = 1$, $A_0 = 0$ and $A_1 = A$. So we can assume that $|A| > \kappa$. Let $\lambda = |A|$. Then $A \cong F/K$, where $F = R^{(\lambda)}$ is a free module. We enumerate the elements of F in a λ–sequence: $F = \{x_\alpha \mid \alpha < \lambda\}$. By induction on α, we will define a sequence $(A_\alpha \mid \alpha \leq \lambda)$ so that for all $\alpha \leq \lambda$, A_α is pure in A and belongs to ${}^\perp\mathcal{C}$. Since each $C \in \mathcal{C}$ is pure–injective, it will follow from the long exact sequence induced by

$$0 \to A_\alpha \to A_{\alpha+1} \to A_{\alpha+1}/A_\alpha \to 0$$

that $A_{\alpha+1}/A_\alpha \in \mathcal{A}$ for all $\alpha < \lambda$.

A_α will be constructed so that it equals $(R^{(I_\alpha)} + K)/K$ for some $I_\alpha \subseteq \lambda$ such that $R^{(I_\alpha)} \cap K$ is pure in K. Let $A_0 = 0$. Assume A_β has been defined for all $\beta < \sigma$. Suppose first that $\sigma = \alpha + 1$. By induction on $n < \omega$ we will define an increasing chain $F_0 \subseteq F_1 \subseteq \dots$ and then put $A_{\alpha+1} = \bigcup_{n<\omega}(F_n + K)/K$. We require that $|F_{n+1}/F_n| \leq \kappa$ for all $n < \omega$, and furthermore: for n odd, that

$(F_n+K)/K$ is pure in F/K; for n even, that $F_n = R^{(J_n)}$ for some $J_n \supseteq J_{n-2} \supseteq \cdots \supseteq J_0$ and $F_n \supseteq K'_n \supseteq K'_{n-2} \supseteq \cdots \supseteq K'_0$ where $F_{n-1} \cap K \subseteq K'_n \subseteq_* K$. (Roughly speaking, the condition for n odd will take care of the purity of $A_{\alpha+1}$ in A, while the condition for n even of $A_{\alpha+1} \in {}^{\perp}\mathcal{C}$.)

First put $F_{-1} = F_0 = R^{(I_\alpha)}$ and let $J_0 = I_\alpha$ and $K'_0 = R^{(I_\alpha)} \cap K$. Assume F_{n-1} has been constructed and n is odd. By Lemma 1.45 (a), there is a pure submodule $(F_n+K)/(F_{n-2}+K) \subseteq_* F/(F_{n-2}+K)$ of cardinality $\leq \kappa$ containing $(x_\alpha R+F_{n-1}+K)/(F_{n-2}+K)$. Moreover, we can choose F_n so that $|F_n/F_{n-1}| \leq \kappa$. By Lemma 1.45 (b), $(F_n+K)/K$ is pure in F/K.

Assume $n > 0$ is even. We first define K'_n: by Lemma 1.45 (a), we find a pure submodule $K'_n/K'_{n-2} \subseteq_* K/K'_{n-2}$ of cardinality $\leq \kappa$ containing $(F_{n-1} \cap K)/K'_{n-2}$. This is possible, since $K'_{n-2} \supseteq F_{n-3} \cap K$ and $(F_{n-1} \cap K)/(F_{n-3} \cap K)$ embeds in F_{n-1}/F_{n-3}, so it has cardinality $\leq \kappa$. By Lemma 1.45 (b), we have $K'_n \subseteq_* K$.

We can choose $J_n \subseteq \lambda$ such that $|J_n - J_{n-2}| \leq \kappa$ and $F_{n-1} + K'_n \subseteq R^{(J_n)} = F_n$. This is possible, since $|(F_{n-1} + K'_n)/F_{n-2}| \leq \kappa$; indeed, we have the exact sequence

$$0 \to F_{n-1}/F_{n-2} \to (F_{n-1} + K'_n)/F_{n-2} \to (F_{n-1} + K'_n)/F_{n-1} \to 0,$$

and $(F_{n-1} + K'_n)/F_{n-1} \cong K'_n/(F_{n-1} \cap K)$ has cardinality $\leq \kappa$, because it is a homomorphic image of K'_n/K'_{n-2}.

Now define $A_{\alpha+1} = \bigcup_{n<\omega}(F_n + K)/K$ and $I_{\alpha+1} = \bigcup_{n<\omega} J_{2n}$. By Lemma 1.45 (d), $A_{\alpha+1} \subseteq_* A$. Clearly $|A_{\alpha+1}/A_\alpha| \leq \kappa$.

We have $A_{\alpha+1} \cong F'/K'$, where $F' = \bigcup_{n<\omega} F_{2n}$ and $K' = F' \cap K$. Also, $F' = R^{(I_{\alpha+1})}$ is free, and $K' = \bigcup_{n<\omega} K'_{2n} (= \bigcup_{n<\omega} F_{2n} \cap K)$ is pure in K by construction and Lemma 1.45 (d).

Let $C \in \mathcal{C}$. In order to prove that $\mathrm{Ext}(A_{\alpha+1}, C) = 0$, we have to extend any $f \in \mathrm{Hom}(K', C)$ to an element of $\mathrm{Hom}(F', C)$. First f extends to K, since $K' \subseteq_* K$ and C is pure–injective. By the assumption (a), we can extend further to F, and then restrict to F'.

Finally, if $\sigma \leq \lambda$ is a limit ordinal, let $A_\sigma = \bigcup_{\beta<\sigma} A_\beta$. Then A_σ has the desired properties by Lemma 1.30 and Lemma 1.45 (d).

(b) implies (a): by Lemma 1.30. ∎

Lemma 1.47. *Let R be a ring, κ be a cardinal and $(\mathcal{A}, \mathcal{B})$ a cotorsion pair. Assume that each $A \in \mathcal{A}$ is a union of a continuous chain, $(A_\alpha \mid \alpha < \sigma)$ of submodules of A such that $A_{\alpha+1}/A_\alpha \in \mathcal{A}$ and $|A_{\alpha+1}/A_\alpha| \leq \kappa$, for all $\alpha+1 < \sigma$. Let $\mathcal{S}$ be a representative set of those elements of $\mathcal{A}$ which have cardinality $\leq \kappa$. Then $\mathcal{B} = \mathcal{S}^\perp$.*

Proof. Clearly $\mathcal{B} \subseteq \mathcal{S}^\perp$. Conversely, take $N \in \mathcal{S}^\perp$. Let $A \in \mathcal{A}$. By assumption, and by the choice of $\mathcal{S}$, $\mathrm{Ext}^1_R(A_{\alpha+1}/A_\alpha, N) = 0$ for all $\alpha < \sigma$. By Lemma 1.30, $\mathrm{Ext}(A, N) = 0$, so $N \in \mathcal{B}$, and $\mathcal{B} = \mathcal{S}^\perp$. ∎

Theorem 1.48. *Let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair cogenerated by a class $\mathcal{C} \subseteq \mathcal{PI}$. Then $\mathfrak{C}_\mathcal{C}$ is complete, closed, and hence perfect.*

Proof. Let $\kappa = |R| + \aleph_0$. Denote by $\mathcal{S}$ the direct sum of a representative set of the class $\{A \in \text{Mod–}R \mid |A| \leq \kappa \text{ and } \mathrm{Ext}(A, \mathcal{C}) = 0\}$.

By Lemma 1.46, each $A \in \mathcal{A}$ has a κ–refinement $(A_\alpha \mid \alpha \leq \lambda)$. By Lemma 1.47, $\mathfrak{C}_\mathcal{C}$ is generated by a set, so $\mathfrak{C}_\mathcal{C}$ is a complete cotorsion pair by Theorem 1.40 (b).

Let C be a pure–injective module. By Lemma 1.35, ${}^\perp C$ is closed under direct limits. It follows that the cotorsion pair $\mathfrak{C}_\mathcal{C}$ is closed. Finally, $\mathfrak{C}_\mathcal{C}$ is perfect by Corollary 1.28. ■

Theorem 1.48 can be extended to higher Ext–orthogonal classes with the help of the following result due to Auslander:

Lemma 1.49. *Let R be a ring. Then the class $\mathcal{PI}$ is cosyzygy closed.*

Proof. Let
$$0 \to P \xrightarrow{\mu} E \xrightarrow{\pi} F \to 0 \tag{1.1}$$
be a short exact sequence, where P is pure–injective and $E = E(P)$ is the injective hull of P. We prove that F is pure–injective. Then any cosyzygy module of P is isomorphic to $F \oplus I$, where I is injective, so $\mathcal{PI}$ is cosyzygy closed.

Applying the functor $(-)^{cc}$ to (1.1), we get the following commutative diagram
$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \xrightarrow{\mu} & E & \xrightarrow{\pi} & F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta_P} & & \downarrow{\scriptstyle \eta_E} & & \downarrow{\scriptstyle \eta_F} & & \\ 0 & \longrightarrow & P^{cc} & \xrightarrow{\mu^{cc}} & E^{cc} & \xrightarrow{\pi^{cc}} & F^{cc} & \longrightarrow & 0, \end{array}$$
where ηs are the evaluation monomorphisms. Since P is pure–injective, η_P splits, so the exists $\rho \in \mathrm{Hom}_R(P^{cc}, P)$ such that $\rho\eta_P = \mathrm{id}_P$. Since E is injective, there is $\sigma \in \mathrm{Hom}_R(E^{cc}, E)$ with $\mu\rho = \sigma\mu^{cc}$. Then $\sigma\eta_E\mu = \sigma\mu^{cc}\eta_P = \mu\rho\eta_P = \mu$. Since μ is minimal, $\varphi = \sigma\eta_E$ is an automorphism.

Denote by $\tau \in \mathrm{Hom}_R(F^{cc}, F)$ the morphism induced by σ. Then $\pi\sigma = \tau\pi^{cc}$. We will prove that $\psi = \tau\eta_F$ is an automorphism.

Let $f \in F$. Take $e \in E$ with $\pi(e) = f$. Then $\psi(f) = \psi\pi(e) = \tau\pi^{cc}\eta_E(e) = \pi\sigma\eta_E(e) = \pi\varphi(e)$.

Since $\pi\varphi$ is surjective, there is $e' \in E$ with $f = \pi\varphi(e')$. Then $\psi(f') = f$, where $f' = \pi(e)$. This proves that ψ is surjective.

Assume that $\psi(f) = 0$. Then $\varphi(e) = \mu(p)$ for some $p \in P$, and $\mu(p) = \varphi\mu(p)$. Since φ is monic, we have $e = \mu(p)$, so $f = \pi(e) = 0$. This shows that ψ is monic.

Finally, since ψ is an automorphism, η_F splits, and F is pure–injective. ■

Corollary 1.50. *Let R be a ring and C be a pure–injective module. Then the class ${}^{\perp_\infty} C$ is closed under pure submodules, pure–epimorphic images and direct limits.*

Proof. Let $\mathcal{E} : 0 \to X \to Y \to Z \to 0$ be a pure–exact sequence with $Y \in {}^{\perp_\infty} C$. Let C_i be the i–th cosyzygy in some injective coresolution of C. By Lemma 1.49, C_i is pure–injective for each $i < \omega$. Since $Y \in {}^\perp C_i$ and $\mathcal{E}$ is pure–exact, we have $Z \in {}^\perp C_i$. So ${}^{\perp_\infty} C$ is closed under pure–epimorphic images. By Lemma 1.35, ${}^{\perp_\infty} C$ is also closed under direct limits.

Finally, from the long exact sequence
$$\ldots \to 0 \to \mathrm{Ext}^n_R(X, C) \to \mathrm{Ext}^{n+1}_R(Z, C) \to 0 \to \ldots$$

we infer that ${}^{\perp_\infty}C$ is closed under pure submodules. ∎

Corollary 1.51. *Let R be a ring, $0 < n < \omega$, and let $\mathcal{C}$ be a class of pure–injective modules. Then the cotorsion pairs $({}^{\perp_n}\mathcal{C}, ({}^{\perp_n}\mathcal{C})^{\perp})$ and $({}^{\perp_\infty}\mathcal{C}, ({}^{\perp_\infty}\mathcal{C})^{\perp})$ are perfect.*

Proof. By Theorem 1.48 and Lemma 1.49. ∎

There is an analogue of Lemma 1.46 for the Tor–bifunctor:

Lemma 1.52. *Let $\mathcal{C}$ be any class of left R–modules. Let $\kappa = |R| + \aleph_0$. The following conditions are equivalent for any module A:*

(a) $A \in {}^{\intercal}\mathcal{C}$.

(b) *There is a cardinal λ such that A has a κ–refinement $(A_\alpha \mid \alpha \leq \lambda)$ such that $A_{\alpha+1}/A_\alpha \in {}^{\intercal}\mathcal{C}$ for all $\alpha < \lambda$.*

Proof. Put $\mathcal{P} = \{C^c \mid C \in \mathcal{C}\}$. Then $\mathcal{P}$ is a class of pure–injective modules and ${}^{\perp}\mathcal{P} = {}^{\intercal}\mathcal{C}$ by Lemma 1.14. So the assertion follows from Lemma 1.46. ∎

Corollary 1.53. *Let R be a ring. Let $\kappa = |R| + \aleph_0$. Then $|L_{Tor}| \leq 2^{2^\kappa}$.*

Proof. Let $\mathcal{S}$ be a representative set of the class of all modules of cardinality $\leq \kappa$. Clearly $|\mathcal{S}| \leq 2^\kappa$. Let $(\mathcal{A}, \mathcal{B})$ be a Tor–pair. By Lemmas 1.52 and 1.32, there is a subset $\mathcal{T} \subseteq \mathcal{S}$ such that $\mathcal{T}^{\intercal} = \mathcal{B}$. It follows that $|L_{Tor}| \leq 2^{2^\kappa}$. ∎

Theorem 1.54.

(a) *Let $\mathcal{C}$ be any class of left R–modules. Then every module has a ${}^{\intercal}\mathcal{C}$–cover.*

(b) *Let $\mathcal{D}$ be any class consisting of character modules (of left R–modules). Then every module has a ${}^{\perp}\mathcal{D}$–cover.*

Proof. (a) As above, we have $\mathcal{A} = {}^{\intercal}\mathcal{C} = {}^{\perp}\mathcal{P}$, where $\mathcal{P}$ is a class of pure–injective modules. Then every module has an $\mathcal{A}$–cover by Theorem 1.48.

(b) Since any character module is pure–injective, every module has a ${}^{\perp}\mathcal{D}$–cover by Theorem 1.48. ∎

Example 1.55. Let R be an artin algebra over a commutative artinian ring k. Let $\mathcal{M}$ be a class of finitely generated modules. Then every module has a ${}^{\perp}\mathcal{M}$–cover. Indeed, any finitely generated module M is isomorphic to M^{dd}. Here $(-)^d$ denotes the *standard duality* $\mathrm{Hom}_k(-, I)$ where $I = \bigoplus_S E(S)$ and S runs over all simple k–modules. So M is pure–injective, and Theorem 1.48 applies.

In view of the importance of the basic construction in Theorem 1.40, it is natural to ask for its dualization. Unlike the direct limit functor, the inverse limit one is not (right) exact in general.

Despite this problem, surprisingly, many results above do have their counterparts in the dual setting. However, the dual of the basic construction holds true only in a weaker form (for more details, see [76]). A complete dualization is not possible in ZFC: in [36], Eklof and Shelah proved that it is consistent with ZFC + GCH that there is no ${}^{\perp}\mathbb{Z}$–precover of the group $\mathbb{Q}$.

A complete dualization is however possible in particular cases. Here, we present a dual of the Bongartz construction (see Lemma 1.43) following [73]:

Proposition 1.56. *Let R and S be rings. Let $A \in S$–Mod–R and $B \in$ Mod–R. Denote by λ the number of generators of the left S–module $\operatorname{Ext}^1_R(B, A)$. Assume that $\operatorname{Ext}^1_R(A^\lambda, A) = 0$. Then there is a module $C \in$ Mod–R such that*

(a) $\operatorname{Ext}^1_R(C, A) = 0$ *and*

(b) *there is an exact sequence $0 \to A^\lambda \to C \xrightarrow{\pi} B \to 0$ in* Mod–R.
In particular, π is a special ${}^{\perp}A$–precover of B.

Proof. We choose extensions $\mathcal{E}_\alpha = 0 \to A \to E_\alpha \xrightarrow{\rho_\alpha} B \to 0$ $(\alpha < \lambda)$ so that their equivalence classes generate $\operatorname{Ext}^1_R(B, A)$ as a left S–module. Let $0 \to A^\lambda \xrightarrow{\mu} C \to B \to 0$ be the extension obtained by a pullback of the direct product extension $0 \to A^\lambda \to \prod_{\alpha<\lambda} E_\alpha \xrightarrow{\prod \rho_\alpha} B^\lambda \to 0$ and of $\Delta_B \in \operatorname{Hom}_R(B, B^\lambda)$ defined by $\Delta_B(b) = (b \mid \alpha < \lambda)$. For each $\alpha < \lambda$, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A^\lambda \arrow[r] \arrow[d, equal] & \prod_{\alpha<\lambda} E_\alpha \arrow[r, "\prod \rho_\alpha"] & B^\lambda \arrow[r] \arrow[d, "\Delta_B\uparrow\sigma_\alpha"'] & 0 \\
0 \arrow[r] & A^\lambda \arrow[r, "\mu"] \arrow[d, "\pi_\alpha"'] & C \arrow[r] \arrow[u, "\tau"] \arrow[d, "h"'] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & X_\alpha \arrow[r, "g"] & B \arrow[r] \arrow[d, equal] & 0, \\
0 \arrow[r] & A \arrow[r] & E_\alpha \arrow[r, "\rho_\alpha"] & B \arrow[r] & 0,
\end{tikzcd}$$

where σ_α is the α–th projection of B^λ to B, and the third row is obtained by pushing out the second row along the α–th canonical projection π_α of A^λ onto A. Using the α–th projection η_α of $\prod_{\alpha<\lambda} E_\alpha$ onto E_α and the pushout property, we get $\varphi \in \operatorname{Hom}_R(X_\alpha, E_\alpha)$, making the lower left square commutative.

Since $\operatorname{Im}(f) = \operatorname{Ker}(g)$, $\operatorname{Im}(h) + \operatorname{Ker}(g) = X_\alpha$ and $gh = \sigma_\alpha(\prod \rho_\alpha)\tau = \rho_\alpha \eta_\alpha \tau = \rho_\alpha \varphi h$, we infer that also the lower right square is commutative. This means that the third and fourth rows are equivalent as extensions of A by B.

Consider the long exact sequence

$$\begin{aligned} &0 \to \operatorname{Hom}_R(B, A) \to \operatorname{Hom}_R(C, A) \to \operatorname{Hom}_R(A^\lambda, A) \xrightarrow{\delta} \\ &\operatorname{Ext}^1_R(B, A) \to \operatorname{Ext}^1_R(C, A) \xrightarrow{\operatorname{Ext}^1_R(\mu, A)} \operatorname{Ext}^1_R(A^\lambda, A) = 0 \end{aligned}$$

induced by $\operatorname{Ext}^i_R(-, A)$. Since equivalence classes of the extensions $\mathcal{E}_\alpha$ $(\alpha < \lambda)$ generate $\operatorname{Ext}^1_R(B, A)$, the commutative diagram constructed above shows that the connecting S–homomorphism δ is surjective. Hence the S–homomorphism $\operatorname{Ext}^1_R(\mu, A)$ is a monomorphism. This proves that $\operatorname{Ext}^1_R(C, A) = 0$. ∎

2 Deconstruction of cotorsion pairs

By Theorem 1.40, in order to prove that a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is complete, it suffices to show that $\mathfrak{C}$ is generated by a set $\mathcal{S} \subseteq \mathcal{A}$. By Theorem 2.20 below, $\mathfrak{C}$ is generated by a set, if and only if there is a cardinal κ such that each module in $\mathcal{A}$ is $\mathcal{A}^{\leq\kappa}$–filtered. The process of finding $\mathcal{A}^{\leq\kappa}$–filtrations for all modules in $\mathcal{A}$ is called the *deconstruction* of the cotorsion pair $\mathfrak{C}$.

We will start with simple cases of deconstruction that yield approximations in various classes of modules of finite homological dimension.

Then we will proceed with one of the basic general tools of deconstruction, the Hill Lemma. It will be used together with other set–theoretic methods to prove that natural closure properties of the classes $\mathcal{A}$ and $\mathcal{B}$ already imply completeness of the cotorsion pair $\mathfrak{C}$. These results will be crucial for characterizing tilting and cotilting cotorsion pairs in chapter 3.

We start with a proof of the Enochs conjecture saying that any module over any ring has a flat cover. We will also generalize Enochs' construction of torsion–free covers of modules over commutative domains. Both of these results are straightforward consequences of Theorem 1.48 (which in turn is based on the deconstruction Lemma 1.46):

Theorem 2.1. *Let R be a ring.*

(a) *The Enochs cotorsion pair $(\mathcal{FL}, \mathcal{EC})$ is perfect and hereditary. In particular, every module has a flat cover and an Enochs cotorsion envelope.*

(b) *The Warfield cotorsion pair $(\mathcal{TF}, \mathcal{RC})$ is perfect. In particular, every module has a torsion–free cover and a Warfield cotorsion envelope.*

Proof. We have $\mathcal{FL} = {}^{\perp}\mathcal{PI}$, and $\mathcal{TF} = {}^{\perp}\mathcal{D}$, where

$$\mathcal{D} = \{N^c \mid N = R/Rr \text{ and } r \in R\} \subseteq \mathcal{PI}$$

(see Lemma 1.14). So Theorem 1.48 applies to the cotorsion pairs $(\mathcal{FL}, \mathcal{EC})$ and $(\mathcal{TF}, \mathcal{RC})$, respectively. The Enochs cotorsion pair is hereditary by Lemma 1.20. ∎

Example 2.2. Let R be a domain. For any module M, Warfield constructed the *cotorsion hull* of M, that is, an overmodule $\bar{M}$ of M such that $\bar{M} \in \mathcal{RC}$, $\bar{M}/M \in \mathcal{TF}$, and M is a *torsion–free essential submodule* in $\bar{M}$ (that is, there is no non–zero submodule K in $\bar{M}$ such that $M \cap K = 0$ and $\bar{M}/(K + M)$ is torsion–free, cf. [44, XIII.8]). Letting $\mathcal{C} = \mathcal{RC}$, we see that the sequence $\mathcal{E} : 0 \to M \to \bar{M} \to \bar{M}/M \to 0$ is an Ext–generator in the sense of Theorem 1.23. The torsion–free essentiality implies that $\mathcal{E}$ has the property as in Lemma 1.25, so by Lemma 1.26, the $\mathcal{RC}$–envelope of M coincides with the inclusion $M \hookrightarrow \bar{M}$.

Since $\mathcal{FL} = \mathcal{F}_0$, it is natural to consider next the classes $\mathcal{F}_n$ of all modules of flat dimension $\leq n$:

Theorem 2.3. *Let R be a ring and $n \geq 0$. Then the cotorsion pair $(\mathcal{F}_n, (\mathcal{F}_n)^{\perp})$ is perfect and hereditary.*

Proof. If $M, N \in \text{Mod–}R$ and $n \geq 0$, then we have

$$\operatorname{Tor}^R_{n+1}(M,N) \cong \operatorname{Tor}^R_1(\Omega^n(M),N)\,.$$

Since $\mathcal{F}_0 = {}^\perp\mathcal{PI}$, we infer that $\mathcal{F}_n = {}^\perp\mathcal{C}_n$ where $\mathcal{C}_n$ is the class of all n–th cosyzygies of all pure–injective modules. By Lemma 1.49, $\mathcal{C}_n \subseteq \mathcal{PI}$, so Theorem 1.48 applies. Since $\mathcal{F}_n$ is resolving, $(\mathcal{F}_n, (\mathcal{F}_n)^\perp)$ is hereditary by Lemma 1.20. ∎

Another straightforward consequence of the general theory is the existence of special divisible and fp–injective preenvelopes of modules:

Definition 2.4. Let R be a ring and M be a module.

(i) M is *fp–injective*, provided that $\operatorname{Ext}^1_R(F,M) = 0$ for each finitely presented module F. The class of all fp–injective modules is denoted by $\mathcal{FI}$. The two extreme cases are the following: R is right noetherian, iff fp–injectivity coincides with injectivity; and R is von Neumann regular, iff all modules are fp–injective (cf. [2]).

M is *divisible*, provided that $\operatorname{Ext}^1_R(F,M) = 0$ for each cyclically presented module F, that is, $\operatorname{Ext}^1_R(R/rR,M) = 0$ for each $r \in R$. The class of all divisible modules is denoted by $\mathcal{DI}$.

(ii) M is *cp–filtered*, (*fp–filtered*, provided that M is $\mathcal{C}$–filtered where $\mathcal{C}$ is the class of all cyclically presented (finitely presented) modules. Denote by $\mathcal{CF}$ ($\mathcal{FF}$) the class of all cp–filtered (fp–filtered) modules.

(iii) Further, denote by $\mathcal{CS}$ ($\mathcal{FS}$) the class of all direct summands of cp–filtered (fp–filtered) modules.

Sometimes fp–injective modules are called *absolutely pure*, because of the following characterization:

Lemma 2.5. *Let R be a ring and M be a module. Then M is fp–injective, if and only if any embedding $M \subseteq N$ is pure.*

Proof. Assume M is fp–injective and $\nu : M \hookrightarrow N$. Let F be any finitely presented module. Since $\operatorname{Ext}^1_R(F,M) = 0$, the map $\operatorname{Hom}_R(F,\nu)$ is surjective. So ν is a pure embedding.

Conversely, let F be finitely presented and consider the exact sequence $0 \to M \hookrightarrow N \xrightarrow{\pi} F \to 0$. Since M is pure in N, the identity map id_F factors through π. It follows that the sequence splits, so $\operatorname{Ext}^1_R(F,M) = 0$. ∎

Clearly $\mathcal{I}_0 \subseteq \mathcal{FI} \subseteq \mathcal{DI}$ and $\mathcal{CS} \subseteq \mathcal{FS}$ for any ring R. By Lemma 1.33, we have $\mathcal{CS} \subseteq P_1$, when R is a domain. Moreover, $\mathcal{CS} = \mathcal{P}_1$ for any valuation domain by a result of Fuchs (see [44, VI.6]).

Theorem 2.6. *Let R be a ring.*

(a) *$(\mathcal{CS}, \mathcal{DI})$ is a complete cotorsion pair. In particular, every module has a special divisible preenvelope. For any module M, there exist divisible modules D and D', cp–filtered modules C and C', and two exact sequences $0 \to M \to D \to C \to 0$ and $0 \to D' \to C' \to M \to 0$.*

(b) $(\mathcal{FS}, \mathcal{FI})$ *is a complete cotorsion pair. In particular, every module has a special fp–injective preenvelope. For any module M, there exist fp–injective modules I and I', fp–filtered modules F and F', and two exact sequences* $0 \to M \to I \to F \to 0$ *and* $0 \to I' \to F' \to M \to 0$.

Proof. Put $M = \bigoplus_{r \in R} R/rR$ and let N be the direct sum of a representative set of all finitely presented modules. By Theorem 1.40 and Corollary 1.41, $(\mathcal{CS}, \mathcal{DI})$ and $(\mathcal{FS}, \mathcal{FI})$ are complete cotorsion pairs generated by M and N, respectively. The existence of the exact sequences follows from part (b) of Corollary 1.41. ∎

In Corollaries 4.14 and 4.15 below, we will see that the statement of Theorem 2.6 is the best possible in the sense that there exist no divisible envelopes, and no fp–injective envelopes, in general.

We turn to approximations by classes of modules of finite injective and projective dimension:

Theorem 2.7. *Let R be a ring and $n < \omega$. Then $({}^{\perp}\mathcal{I}_n, \mathcal{I}_n)$ is a complete hereditary cotorsion pair. In particular, every module has a special $\mathcal{I}_n$–preenvelope.*

Proof. Let M be a module. Let

$$\mathfrak{I}: \quad 0 \to M \to I_0 \to I_1 \to \dots \to I_{n-1} \to I_n \to \dots$$

be an injective coresolution of M. Let C_n be the n–th cosyzygy of M in $\mathfrak{I}$. Then $M \in \mathcal{I}_n$, iff C_n is injective. By the Baer Criterion, the latter is equivalent to $\operatorname{Ext}^1_R(R/I, C_n) = 0$, and hence – by dimension shifting – to $\operatorname{Ext}^n_R(R/I, M) = 0$, for all right ideals I of R. Denote by S_I the n–th syzygy (in a projective resolution) of the cyclic module R/I. Then $\operatorname{Ext}^n_R(R/I, M) = 0$, iff $\operatorname{Ext}^1_R(S_I, M) = 0$. So $\mathcal{I}_n = (\bigoplus_{I \subseteq R} S_I)^{\perp}$, and the assertion follows by Theorem 1.40 (b).

The cotorsion pair is hereditary by Corollary 1.21 (b), since the class $\mathcal{I}_n$ is cosyzygy closed. ∎

The proof of Theorem 2.7 is based on the existence of a test module for injectivity, that is, on the Baer Criterion. In the dual case, the existence of test modules for projectivity depends on the structure of the base ring. If R is not right perfect, then it is consistent with ZFC + GCH that there are no test modules for projectivity (see [73, §2]). If R is right perfect, then a test module for projectivity always exists:

Lemma 2.8. *Let R be a right perfect ring and $n < \omega$. Let $\mathcal{C}_n = \{\Omega^{-n}(M) \mid M \in \operatorname{simp} R\}$. Then $\mathcal{P}_n = {}^{\perp}\mathcal{C}_n$.*

Proof. Let $n = 0$. We have to prove that $\mathcal{P}_0 = {}^{\perp}(\operatorname{simp} R)$.

Assume $N \in {}^{\perp}(\operatorname{simp} R) \setminus \mathcal{P}_0$. Since R is right perfect, N has a projective cover $0 \to K \hookrightarrow P \to N \to 0$ where K is a non–zero superfluous submodule of P, and K has a maximal submodule L. By assumption, $\operatorname{Ext}^1_R(N, K/L) = 0$. So the projection $\pi : K \to K/L$ can be extended to $\sigma \in \operatorname{Hom}_R(P, K/L)$. Then $\operatorname{Ker} \sigma$ is a maximal submodule of P, so $K \subseteq \operatorname{Rad}(P) \subseteq \operatorname{Ker} \sigma$, and $\pi = \sigma \restriction K = 0$, a contradiction.

Assume that $n > 0$ and consider $N \in \text{Mod–}R$. Then

$$N \in \mathcal{P}_n \iff \Omega^n(N) \in \mathcal{P}_0 \iff \operatorname{Ext}^1_R(\Omega^n(N), \operatorname{simp} R) = 0 \iff$$

$$\mathrm{Ext}^n_R(N, \mathrm{simp}\, R) = 0 \iff \mathrm{Ext}^n_R(N, \mathcal{C}_n) = 0.$$

Thus the lemma follows. ∎

Nevertheless, a result dual to Theorem 2.7 is true for any ring. The proof makes use of a deconstruction of the cotorsion pair $(\mathcal{P}_n, \mathcal{P}_n^\perp)$ resembling Lemma 1.46. The result comes from [1].

Before presenting a proof, we need a generalization of the notion of a right noetherian ring:

Definition 2.9. Let R be a ring. Let κ be a cardinal. Then R is right κ–*noetherian*, provided that each right ideal I of R is $\leq \kappa$–generated. The least infinite cardinal κ such that R is right κ–noetherian is the right *dimension* of R, denoted by $\dim(R)$.

Lemma 2.10. *Let R be a ring, and κ be a cardinal such that $\kappa \geq \dim(R)$. Then any submodule of a $\leq \kappa$–generated module is $\leq \kappa$–generated.*

Proof. First all submodules of cyclic modules are $\leq \dim(R)$–generated, since they are epimorphic images of right ideals. Further, any $\leq \kappa$–generated module M is a union of a continuous chain, $(M_\alpha \mid \alpha \leq \kappa)$, of submodules such that all the factors $M_{\alpha+1}/M_\alpha$ are cyclic. If $K \subseteq M$, then $K \cap M_{\alpha+1}/K \cap M_\alpha$ embeds into $M_{\alpha+1}/M_\alpha$ for each $\alpha < \kappa$, and the assertion follows. ∎

Lemma 2.11. *Let $n < \omega$, R be a ring, $\kappa = \dim(R)$, and $M \in \mathcal{P}_n$. Then M is $\mathcal{P}_n^{\leq\kappa}$–filtered.*

Proof. Let $\lambda = \mathrm{gen}(M) + \kappa$. By Eilenberg's trick, M has a free resolution

$$\mathcal{R} \quad : \quad 0 \to R^{(A_n)} \overset{f_n}{\to} R^{(A_{n-1})} \to \ldots \to R^{(A_1)} \overset{f_1}{\to} R^{(A_0)} \overset{f_0}{\to} M \to 0,$$

such that $|A_i| \leq \lambda$ for each $i \leq n$.

Let $(m_\alpha \mid \alpha < \lambda)$ be a set of R–generators of M. By induction on α, we will construct a $\mathcal{P}_n^{<\kappa}$–filtration $(M_\alpha \mid \alpha < \lambda)$ of M together with free resolutions $\mathcal{R}_\alpha$ of M_α which are restrictions of $\mathcal{R}$:

$$\mathcal{R}_\alpha \quad : \quad 0 \to F_{\alpha,n} \overset{f_n \restriction F_{\alpha,n}}{\to} F_{\alpha,n-1} \to \ldots \to F_{\alpha,1} \overset{f_1 \restriction F_{\alpha,1}}{\to} F_{\alpha 0} \overset{f_0 \restriction F_{\alpha,0}}{\to} M_\alpha \to 0,$$

so that $m_\alpha \in M_{\alpha+1}$, $F_{\alpha,i} = R^{(A_{\alpha,i})}$ for some $A_{\alpha,i} \subseteq A_i$, and $|A_{\alpha+1,i} \setminus A_{\alpha,i}| \leq \kappa$, for all $\alpha < \lambda$ and $i \leq n$.

First $M_0 = 0$ and $A_{0,i} = \emptyset$ for all $i \leq n$. Assume M_α and $\mathcal{R}_\alpha$ are defined. If $M_\alpha \neq M$, let $\gamma < \lambda$ be the least index such that $m_\gamma \notin M_\alpha$. Clearly there is a subset $B_0 \subseteq A_0$ of cardinality $\leq \kappa$ (in fact, a finite one) such that $m_\gamma \subseteq f_0(R^{(A_{\alpha,0} \cup B_0)})$.

Since

$$\mathrm{Ker}(f_0 \restriction R^{(A_{\alpha,0})}) = \mathrm{Ker}(f_0 \restriction R^{(A_{\alpha,0} \cup B_0)}) \cap R^{(A_{\alpha,0})},$$

we have

$$\mathrm{Ker}(f_0 \restriction R^{(A_{\alpha,0} \cup B_0)}) / \mathrm{Ker}(f_0 \restriction R^{(A_{\alpha,0})}) \cong$$
$$\cong (R^{(A_{\alpha,0})} + \mathrm{Ker}(f_0 \restriction R^{(A_{\alpha,0} \cup B_0)}))/R^{(A_{\alpha,0})}.$$

The latter module is a submodule in $R^{(A_{\alpha,0} \cup B_0)}/R^{(A_{\alpha,0})} \cong R^{(B_0)}$. So the exactness of $\mathcal{R}_\alpha$ at $F_{\alpha,0}$, of $\mathcal{R}$ at $R^{(A_0)}$, and Lemma 2.10 yield the existence of a subset $B_1 \subseteq A_1$ of cardinality $\leq \kappa$ such that $\operatorname{Ker}(f_0 \restriction R^{(A_{\alpha,0} \cup B_0)}) \subseteq f_1(R^{(A_{\alpha,1} \cup B_1)})$. Similarly, there is a subset $B_2 \subseteq A_2$ of cardinality $\leq \kappa$ such that $\operatorname{Ker}(f_1 \restriction R^{(A_{\alpha,1} \cup B_1)}) \subseteq f_2(R^{(A_{\alpha,2} \cup B_2)})$, etc. Finally, there is a subset $B_n \subseteq A_n$ of cardinality $\leq \kappa$ such that $\operatorname{Ker}(f_{n-1} \restriction R^{(A_{\alpha,n-1} \cup B_{n-1})}) \subseteq f_n(R^{(A_{\alpha,n} \cup B_n)})$.

Now there is a subset $B_{n-1} \subseteq B'_{n-1} \subseteq A_{n-1}$ of cardinality $\leq \kappa$ such that $f_n(R^{(A_{\alpha,n} \cup B_n)}) \subseteq R^{(A_{\alpha,n-1} \cup B'_{n-1})}$, etc. Finally, there is a subset $B_0 \subseteq B'_0 \subseteq A_0$ of cardinality $\leq \kappa$ such that $f_1(R^{(A_{\alpha,1} \cup B'_1)}) \subseteq R^{(A_{\alpha,0} \cup B'_0)}$.

Continuing this back and forth procedure in $\mathcal{R}$, we obtain, for each $i \leq n$, a countable chain $B_i \subseteq B'_i \subseteq B_i'' \subseteq \dots$ consisting of subsets of A_i of cardinality $\leq \kappa$. Let $C_i = B_i \cup B'_i \cup B_i'' \cup \dots$. Then C_i has cardinality $\leq \kappa$, the sequence

$$\begin{aligned} \mathcal{R}_{\alpha+1} \ : \ 0 \to F_{\alpha+1,n} &\overset{f_n \restriction F_{\alpha+1,n}}{\to} F_{\alpha+1,n-1} \to \dots \\ &\dots \to F_{\alpha+1,1} \overset{f_1 \restriction F_{\alpha+1,1}}{\to} F_{\alpha+1,0} \overset{f_0 \restriction F_{\alpha+1,0}}{\to} N \to 0, \end{aligned}$$

with $F_{\alpha+1,i} = R^{(A_{\alpha,i} \cup C_i)}$ is exact, and $\{m_\gamma\} \cup M_\alpha \subseteq N$. (The backward procedure takes care of kernels being inside images, while the forward one of the resulting sequence being a complex.)

We put $M_{\alpha+1} = N$. Note that $\mathcal{R}_\alpha$ is an exact subcomplex of the exact complex $\mathcal{R}_{\alpha+1}$, so the factor complex $\mathcal{R}_{\alpha+1}/\mathcal{R}_\alpha$ is exact. This shows that $M_{\alpha+1}/M_\alpha \in \mathcal{P}_n^{\leq \kappa}$.

For a limit ordinal $\alpha < \lambda$, we define $A_{\alpha,i} = \bigcup_{\beta<\alpha} A_{\beta,i}$ and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. Then the corresponding restriction of $\mathcal{R}$ is a free resolution of M_α. ∎

Now we easily derive

Theorem 2.12. *Let R be a ring and $n < \omega$.*

(a) *Then $\mathfrak{C}_n = (\mathcal{P}_n, \mathcal{P}_n^\perp)$ is a complete hereditary cotorsion pair. In particular, every module has a special $\mathcal{P}_n$–precover.*

(b) *If R is right $\aleph_0$–noetherian, then $\mathfrak{C}_n$ is generated by (a representative set of) the class $\mathcal{P}_n^{\leq \omega}$.*

(c) *If R is right perfect, then the cotorsion pair $\mathfrak{C}_n$ is perfect.*

Proof. Let $\kappa = \dim(R)$. By Lemmas 1.30 and 2.11, we have $\mathcal{P}_n^\perp = (\mathcal{P}_n^{\leq\kappa})^\perp$. Clearly $\mathcal{P}_n^{\leq\kappa}$ has a representative set of elements. By Corollary 1.41 (b) and Lemma 1.33, we get ${}^\perp(\mathcal{P}_n^\perp) = \mathcal{P}_n$, so $\mathfrak{C}_n$ is a complete cotorsion pair. $\mathfrak{C}_n$ is hereditary by Corollary 1.21 (a). If R is right perfect, then $\mathcal{P}_n = \mathcal{F}_n$, and Theorem 2.3 applies. ∎

Though $(\mathcal{P}_n, \mathcal{P}_n^\perp)$ is always complete, there may be no minimal approximations available if R is not right perfect. This is well–known for $n = 0$ (see [17]). For the case $n = 1$, see Corollary 4.15 below.

Theorem 2.13. *Let R be a right noetherian ring. Then $\mathfrak{E}_n = ({}^\perp(\mathcal{I}_n^\perp), \mathcal{I}_n^\perp)$ is a complete cotorsion pair. Moreover, if $\operatorname{inj\,dim} R \leq n$, then $\mathfrak{E}_n = (\mathcal{I}_n, \mathcal{I}_n^\perp)$ is a perfect cotorsion pair.*

Proof. Since R is right noetherian, there is a cardinal κ such that each injective module is a direct sum of $\leq \kappa$ generated modules by the Faith–Walker Theorem. So an analogue of Lemma 2.11 holds true for $\mathcal{I}_n$ – there is a set $\mathcal{S} \subset \mathcal{I}_n$ such that $\mathcal{S}^\perp = \mathcal{I}_n^\perp$ (the proof is dual to the one given in Lemma 2.11, via a back and forth procedure in an injective coresolution of an element of $\mathcal{I}_n$). By Theorem 1.40 (b) and Lemma 1.30, it follows that $\mathfrak{C}_n$ is complete.

By Corollary 1.36, $\mathcal{I}_n$ is closed under direct limits. Assume $R \in \mathcal{I}_n$. Then w.l.o.g. $R \in \mathcal{S}$, so the class ${}^\perp(\mathcal{I}_n^\perp)$ consists of direct summands of $\mathcal{S}$–filtered modules by Corollary 1.42. By induction on the length of the $\mathcal{S}$–filtration, we get that ${}^\perp(\mathcal{I}_n^\perp) = \mathcal{I}_n$. Finally, $\mathfrak{C}_n$ is perfect by Corollary 1.28. ∎

Example 2.14. Let R be an *Iwanaga–Gorenstein ring*, that is, a left and right noetherian ring with finite injective dimension on either side. Then the left and the right injective dimensions of R coincide with some $n < \omega$, and R is called *n–Iwanaga–Gorenstein.* Moreover, all (left or right) R–modules of finite injective (projective, flat) dimension have injective (projective, flat) dimension $\leq n$, so in Mod–R, we have $\mathcal{P} = \mathcal{P}_n = \mathcal{I} = \mathcal{I}_n = \mathcal{F}_n$ (see e.g. [40, §9.1]). The latter is a covering class in Mod–R by Theorem 2.3. The same is true for the corresponding classes of left R–modules: we will denote by $\mathcal{L}$ the class of all left R–modules of finite projective dimension.

A module M is *Gorenstein projective (Gorenstein injective, Gorenstein flat)*, if $M \in {}^\perp\mathcal{P}$ ($M \in \mathcal{P}^\perp$ and $M \in {}^\intercal\mathcal{L}$). Denote by $\mathcal{GP}$ ($\mathcal{GI}$, $\mathcal{GF}$) the classes of all Gorenstein projective (injective, flat) modules. By Theorem 2.7, $(\mathcal{GP}, \mathcal{P})$ is a complete hereditary cotorsion pair, while $(\mathcal{P}, \mathcal{GI})$ is a perfect hereditary cotorsion pair by Theorem 2.13, and $(\mathcal{GF}, \mathcal{L})$ is a Tor–pair. In particular, every module has a Gorenstein injective envelope, and a Gorenstein flat cover. (In Section 5 we will prove that $(\mathcal{P}, \mathcal{GI})$ is actually a tilting cotorsion pair. This will yield the validity of the first finitistic dimension conjecture for R.)

Similarly, one defines modules of Gorenstein projective (injective, flat) dimension $\leq m$, and proves the existence of the corresponding cotorsion pairs, envelopes and covers. For more details, we refer to [40, Chapters 9-11].

Assume $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a perfect cotorsion pair. Then often the modules in the kernel $\mathcal{K}$ of $\mathfrak{C}$ can be classified up to isomorphism by cardinal invariants (see e.g. Theorem 2.47 below). There are two ways of extending this classification:

(i) Any module $A \in \mathcal{A}$ determines — by an iteration of $\mathcal{B}$–envelopes (of A, of the cokernel of the $\mathcal{B}$–envelope of A, etc.) — a long exact sequence all of whose members (except for A) belong to $\mathcal{K}$. This sequence is called the *minimal $\mathcal{B}$–coresolution* of A. The sequence of the cardinal invariants of the modules from $\mathcal{K}$ occurring in the coresolution is an invariant of A. In this way the structure theory of the modules in $\mathcal{K}$ is extended to a structure theory for $\mathcal{A}$.

(ii) Dually, any module $B \in \mathcal{B}$ determines — by an iteration of $\mathcal{A}$–covers — a long exact sequence all of whose members (except for B) belong to $\mathcal{K}$, the *minimal $\mathcal{A}$–resolution* of B. This yields a sequence of cardinal invariants for any module $B \in \mathcal{B}$.

For more specific examples of (i) and (ii), we consider the case when R is a commutative noetherian ring:

If $\mathfrak{C} = (\text{Mod–}R, \mathcal{I}_0)$, then $\mathcal{K} = \mathcal{I}_0$, and by the classical theory of Matlis, each $M \in \mathcal{K}$ is determined up to isomorphism by the multiplicities of indecomposable injectives $E(R/p)$ (p a prime ideal of R) occurring in an indecomposable decomposition of M. The cardinal invariants of arbitrary modules (in $\mathcal{A} = $ Mod–R) constructed in (i) are called the *Bass invariants*.

A formula for their computation goes back to Bass: the multiplicity of $E(R/p)$ in the m–th term of the minimal injective coresolution of a module N is

$$\mu_m(p, N) = \dim_{k(p)} \operatorname{Ext}^m_{R_{(p)}}(k(p), N_{(p)}),$$

where $k(p) = R_{(p)}/\operatorname{Rad}(R_{(p)})$ is the residue field, and $R_{(p)}$ and $N_{(p)}$ are the localizations of R and N at p, respectively, for all $p \in \operatorname{spec} R$ and $m \geq 0$ (see [40, §9.2]).

If $\mathfrak{C} = (\mathcal{F}_0, \mathcal{EC})$, then $\mathcal{K}$ consists of the flat pure–injective modules M. These are described by the ranks of the free modules F_p over the localizations $R_{(p)}$ whose p–completions occur in the decomposition of M for $0 \neq p \in \operatorname{spec} R$, and by the rank of the free module F_0 over $R_{(0)}$ that occurs in the decomposition of M in case $0 \in \operatorname{spec} R$, see [40, §5.3].

The construction (ii) then yields a sequence of invariants for any Enochs cotorsion module N. These invariants are called the *dual Bass invariants*. A formula for their computation is due to Xu [79, §5.2]: the rank of F_p in the m–th term of the minimal flat resolution of N is

$$\pi_m(p, N) = \dim_{k(p)} \operatorname{Tor}^{R_{(p)}}_m(k(p), \operatorname{Hom}_R(R_{(p)}, N)),$$

where $p \in \operatorname{spec} R$ and $m \geq 0$.

Now, we present Hill's construction of large families of submodules starting from a single continuous chain. Our presentation is based on [72], [68], and [42].

We start with fixing our notation:

Definition 2.15. Let R be a ring and $\mathcal{M}$ be a continuous chain of modules, $(M_\alpha \mid \alpha \leq \sigma)$. Consider a family of modules $(A_\alpha \mid \alpha < \sigma)$ such that $M_{\alpha+1} = M_\alpha + A_\alpha$ for each $\alpha < \sigma$.
A subset S of σ is *closed*, if every $\beta \in S$ satisfies

$$M_\beta \cap A_\beta \subseteq \sum_{\alpha \in S, \alpha < \beta} A_\alpha.$$

The *height*, $\operatorname{hgt}(x)$, of an element $x \in M_\sigma$ is defined as the least ordinal $\alpha < \sigma$ such that $x \in M_{\alpha+1}$. For any subset S of σ, we define $M(S) = \sum_{\alpha \in S} A_\alpha$.

For each ordinal $\alpha \leq \sigma$, we have $M_\alpha = \sum_{\beta < \alpha} A_\beta$, so α $(= \{\beta < \sigma \mid \beta < \alpha\})$ is a closed subset of σ.

Lemma 2.16. *Let S be a closed subset of σ, and $x \in M(S)$. Let $S' = \{\alpha \in S \mid \alpha \leq hgt(x)\}$. Then $x \in M(S')$.*

Proof. Let $x \in M(S)$. Then $x = x_1 + \cdots + x_k$ where $x_i \in A_{\alpha_i}$ for some $\alpha_i \in S$, $1 \leq i \leq k$. W.l.o.g., $\alpha_1 < \cdots < \alpha_k$, and α_k is minimal.

If $\alpha_k > \operatorname{hgt}(x)$, then $x_k = x - x_1 - \cdots - x_{k-1} \in M_{\alpha_k} \cap A_{\alpha_k} \subseteq \sum_{\alpha \in S, \alpha < \alpha_k} A_\alpha$, since S is closed, in contradiction with the minimality of α_k. ∎

As an immediate corollary, we have

Corollary 2.17. *Let S be a closed subset of σ, and $x \in M(S)$. Then $hgt(x) \in S$.*

Lemma 2.18. *Let $(S_i \mid i \in I)$, be a family of closed subsets of σ. Then*

$$M(\bigcap_{i\in I} S_i) = \bigcap_{i\in I} M(S_i).$$

Proof. Let $T = \bigcap_{i\in I} S_i$. Clearly $M(T) \subseteq \bigcap_{i\in I} M(S_i)$. Suppose there is an $x \in \bigcap_{i\in I} M(S_i)$ such that $x \notin M(T)$, and choose such an x of minimal height. Then $x = y + z$ for some $y \in A_{\mathrm{hgt}(x)}$ and $z \in M_{\mathrm{hgt}(x)}$. By Corollary 2.17, $\mathrm{hgt}(x) \in S_i$ for all $i \in I$, so $\mathrm{hgt}(x) \in T$, and $y \in M(T)$. Then $z \in \bigcap_{i\in I} M(S_i)$, $z \notin M(T)$ and $\mathrm{hgt}(z) < \mathrm{hgt}(x)$, in contradiction to minimality. ■

Next we prove that intersections and unions of closed subsets are again closed:

Proposition 2.19. *Let $(S_i \mid i \in I)$ be a family of closed subsets of σ. Then both the union and the intersection of this family are again closed in σ. That is, closed subsets of σ form a complete sublattice of 2^σ.*

Proof. As for the union, if $\beta \in S = \bigcup_{i\in I} S_i$, then $\beta \in S_i$ for some $i \in I$, and $M_\beta \cap A_\beta \subseteq \sum_{\alpha\in S_i,\alpha<\beta} A_\alpha \subseteq \sum_{\alpha\in S,\alpha<\beta} A_\alpha$.

For the intersection, let $\beta \in T = \bigcap_{i\in I} S_i$. Then $M_\beta \cap A_\beta \subseteq M(S_i \cap \beta)$ for each $i \in I$. Therefore Lemma 2.18 implies that $M_\beta \cap A_\beta \subseteq \bigcap_{i\in I} M(S_i \cap \beta) = M(T \cap \beta)$ which exactly says that T is closed. ■

The general version of the Hill Lemma can now be stated as follows:

Theorem 2.20. *Let R be a ring, κ an infinite regular cardinal and $\mathcal{C}$ a set of $< \kappa$–presented modules. Let M be a module with a $\mathcal{C}$–filtration $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$. Then there is a family $\mathcal{F}$ consisting of submodules of M such that:*

(H1) $\mathcal{M} \subseteq \mathcal{F}$.

(H2) $\mathcal{F}$ is closed under arbitrary sums and intersections.

(H3) Let $N, P \in \mathcal{F}$ be such that $N \subseteq P$. Then there exists a $\mathcal{C}$–filtration $(\bar{P}_\gamma \mid \gamma \leq \tau)$ of the module $\bar{P} = P/N$ such that $\tau \leq \sigma$, and for each $\gamma < \tau$ there is a $\beta < \sigma$ with $\bar{P}_{\gamma+1}/\bar{P}_\gamma$ isomorphic to $M_{\beta+1}/M_\beta$.

(H4) Let $N \in \mathcal{F}$ and X be a subset of M of cardinality $< \kappa$. Then there is a $P \in \mathcal{F}$ such that $N \cup X \subseteq P$ and P/N is $< \kappa$–presented.

Proof. Consider a family of $< \kappa$–generated modules $(A_\alpha \mid \alpha < \sigma)$ such that for each $\alpha < \sigma$:

$$M_{\alpha+1} = M_\alpha + A_\alpha,$$

as in Definition 2.15. We claim that

$$\mathcal{F} = \{M(S) \mid S \text{ a closed subset of } \sigma\}$$

has properties (H1)–(H4).

Property (H1) is clear, since each ordinal $\alpha \leq \sigma$ is a closed subset of σ. Property (H2) follows by Proposition 2.19 and Lemma 2.18.

Property (H3) is proved as follows: we have $N = M(S)$ and $P = M(T)$ for some closed subsets S, T. Since $S \cup T$ is closed, we can w.l.o.g. assume that $S \subseteq T$. For each $\beta \leq \sigma$, put

$$F_\beta = N + \sum_{\alpha \in T \setminus S, \alpha < \beta} A_\alpha = M(S \cup (T \cap \beta)) \qquad \text{and} \qquad \bar{F}_\beta = F_\beta / N.$$

Clearly $(\bar{F}_\beta \mid \beta \leq \sigma)$ is a filtration of $\bar{P} = P/N$ such that $\bar{F}_{\beta+1} = \bar{F}_\beta + (A_\beta + N)/N$ for $\beta \in T \setminus S$ and $\bar{F}_{\beta+1} = \bar{F}_\beta$ otherwise. Let $\beta \in T \setminus S$. Then

$$\bar{F}_{\beta+1}/\bar{F}_\beta \cong F_{\beta+1}/F_\beta \cong A_\beta/(F_\beta \cap A_\beta),$$

and

$$F_\beta \cap A_\beta \supseteq (\sum_{\alpha \in T, \alpha < \beta} A_\alpha) \cap A_\beta = M_\beta \cap A_\beta.$$

However, if $x \in F_\beta \cap A_\beta$, then $\operatorname{hgt}(x) \leq \beta$, so $x \in M(T')$ by Lemma 2.16, where $T' = \{\alpha \in S \cup (T \cap \beta) \mid \alpha \leq \beta\}$. By Proposition 2.19, we get $x \in M_\beta$ because $\beta \notin S$. Hence $F_\beta \cap A_\beta = M_\beta \cap A_\beta$ and $\bar{F}_{\beta+1}/\bar{F}_\beta \cong A_\beta/(M_\beta \cap A_\beta) \cong M_{\beta+1}/M_\beta$. The $\mathcal{C}$–filtration $(\bar{P}_\gamma \mid \gamma \leq \tau)$ is obtained from $(\bar{F}_\beta \mid \beta \leq \sigma)$ by removing possible repetitions, and (H3) follows. Denote by τ' the ordinal type of the well–ordered set $(T \setminus S, <)$. Notice that the length τ of the filtration can be taken as $1 + \tau'$ (the ordinal sum, hence $\tau = \tau'$ for τ' infinite).

For property (H4), we first prove that every subset of σ of cardinality $< \kappa$ is contained in a closed subset of cardinality $< \kappa$. Because κ is an infinite regular cardinal, by Proposition 2.19, it is enough to prove this only for one–element subsets of σ. So we prove that every $\beta < \sigma$ is contained in a closed subset of cardinality $< \kappa$, by induction on β. For $\beta < \kappa$, we just take $S = \beta + 1$. Otherwise, the short exact sequence

$$0 \to M_\beta \cap A_\beta \to A_\beta \to M_{\beta+1}/M_\beta \to 0$$

shows that $M_\beta \cap A_\beta$ is $< \kappa$–generated. Thus $M_\beta \cap A_\beta \subseteq \sum_{\alpha \in S_0} A_\alpha$ for a subset $S_0 \subseteq \beta$ of cardinality $< \kappa$. Moreover, we can assume that S_0 is closed in σ by inductive premise, and put $S = S_0 \cup \{\beta\}$. To show that S is closed, it suffices to check the definition only for β. But $M_\beta \cap A_\beta \subseteq M(S_0) = \sum_{\alpha \in S, \alpha < \beta} A_\alpha$.

Finally, let $N = M(S)$ where S is closed in σ, and let X be a subset of M of cardinality $< \kappa$. Then $X \subseteq \sum_{\alpha \in T} A_\alpha$ for a subset T of σ of cardinality $< \kappa$. By the preceding paragraph, we can assume that T is closed in σ. Let $P = M(S \cup T)$. Then P/N is $\mathcal{C}$–filtered by property (H3), and the filtration can be chosen indexed by 1+ the ordinal type of $T \setminus S$, which is certainly less than κ. In particular, P/N is $< \kappa$–presented. ∎

There is also a rank version of the Hill Lemma for torsion–free modules over domains. Given a domain R and a torsion–free module M, we define the *rank*, $\operatorname{rk} X$, of a subset $X \subseteq M$ as the torsion–free rank of the submodule $\langle X \rangle$ of M generated by X. Note that $\operatorname{rk} X \leq |X|$.

Theorem 2.21. *Let R be a domain, κ an infinite regular cardinal and $\mathcal{C}$ a set of torsion–free R–modules. Let M be a module with a $\mathcal{C}$–filtration $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$. Assume moreover that for each $\alpha < \sigma$ there is a submodule A_α of M of rank $< \kappa$ such that $M_{\alpha+1} = M_\alpha + A_\alpha$.*

Then there is a family $\mathcal{F}$ of submodules of M such that the properties (H1), (H2) and (H3) from Theorem 2.20 hold true. Moreover, the following rank version of property (H4) holds:

(H4*) *Let $N \in \mathcal{F}$ and X be a subset of M with rk$X < \kappa$. Then there are $P \in \mathcal{F}$ and a submodule $A \subseteq M$ of rank $< \kappa$ such that $N \cup X \subseteq P$ and $P = N + A$.*

For a proof of Theorem 2.21 we refer to [72].

By Corollary 1.42, if $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a cotorsion pair generated by a set $\mathcal{C}$ containing R, then $\mathcal{A}$ coincides with the class of all direct summands of $\mathcal{C}$–filtered modules. Our next goal is to remove the term "direct summands" in this characterization of $\mathcal{A}$ on account of replacing the set $\mathcal{C}$ by a suitable small subset of $\mathcal{A}$, with the help of Theorem 2.20:

Lemma 2.22. *Let κ be an uncountable regular cardinal and $\mathcal{C}$ a set of $< \kappa$–presented modules. Denote by $\mathcal{A}$ the class of all direct summands of $\mathcal{C}$–filtered modules. Then every module in $\mathcal{A}$ is $\mathcal{A}^{<\kappa}$–filtered.*

Proof. Let $K \in \mathcal{A}$, so there is a $\mathcal{C}$–filtered module M such that $M = K \oplus L$ for some $L \subseteq M$. Denote by $\pi_K : M \to K$ and $\pi_L : M \to L$ the corresponding projections. Let $\mathcal{F}$ be the family of submodules of M as in Theorem 2.20. We proceed in two steps:

Step I: By induction, we construct a continuous chain, $(N_\alpha \mid \alpha \leq \tau)$, of submodules of M such that $N_\tau = M$ and

(a) $N_\alpha \in \mathcal{F}$,

(b) $N_\alpha = \pi_K(N_\alpha) + \pi_L(N_\alpha)$, and

(c) $N_{\alpha+1}/N_\alpha$ is $< \kappa$–presented

for each $\alpha < \tau$.

First $N_0 = 0$ and $N_\beta = \bigcup_{\alpha<\beta} N_\alpha$ for all limit ordinals $\beta \leq \tau$. Suppose we have $N_\alpha \subsetneqq M$ and we wish to construct $N_{\alpha+1}$. Take $x \in M \setminus N_\alpha$; by property (H4), there is $Q_0 \in \mathcal{F}$ such that $N_\alpha \cup \{x\} \subseteq Q_0$ and Q_0/N_α is $< \kappa$–presented. Let X_0 be a subset of Q_0 of cardinality $< \kappa$ such that the set $\{x + N_\alpha \mid x \in X_0\}$ generates Q_0/N_α. Put $Z_0 = \pi_K(Q_0) \oplus \pi_L(Q_0)$. Clearly $Q_0/N_\alpha \subseteq Z_0/N_\alpha$. Since $\pi_K(N_\alpha), \pi_L(N_\alpha) \subseteq N_\alpha$, the module Z_0/N_α is generated by the set

$$\{x + N_\alpha \mid x \in \pi_K(X_0) \cup \pi_L(X_0)\}.$$

Thus we can find $Q_1 \in \mathcal{F}$ such that $Z_0 \subseteq Q_1$ and Q_1/N_α is $< \kappa$–presented. Similarly, we infer that Z_1/N_α is $< \kappa$–generated for $Z_1 = \pi_K(Q_1) \oplus \pi_L(Q_1)$, and find $Q_2 \in \mathcal{F}$ with $Z_1 \subseteq Q_2$ and Q_2/N_α a $< \kappa$–presented module. In this way we obtain a chain $Q_0 \subseteq Q_1 \subseteq \dots$ such that for all $i < \omega$: $Q_i \in \mathcal{F}$, Q_i/N_α is $< \kappa$–presented, and $\pi_K(Q_i) + \pi_L(Q_i) \subseteq Q_{i+1}$. It is easy to see that $N_{\alpha+1} = \bigcup_{i<\omega} Q_i$ satisfies the properties (a)–(c).

Step II: By condition (b), we have

$$\pi_K(N_{\alpha+1}) + N_\alpha = \pi_K(N_{\alpha+1}) \oplus \pi_L(N_\alpha)$$

and similarly for L. Hence

$$\begin{aligned}(\pi_K(N_{\alpha+1}) + N_\alpha) &\cap (\pi_L(N_{\alpha+1}) + N_\alpha)\\ &= (\pi_K(N_{\alpha+1}) \oplus \pi_L(N_\alpha)) \cap (\pi_L(N_{\alpha+1}) \oplus \pi_K(N_\alpha))\\ &= \big(\pi_K(N_{\alpha+1}) \cap (\pi_L(N_{\alpha+1}) \oplus \pi_K(N_\alpha))\big) \oplus \pi_L(N_\alpha)\\ &\qquad\qquad = \pi_K(N_\alpha) \oplus \pi_L(N_\alpha) = N_\alpha\end{aligned}$$

and

$$N_{\alpha+1}/N_\alpha = (\pi_K(N_{\alpha+1}) + N_\alpha)/N_\alpha \oplus (\pi_L(N_{\alpha+1}) + N_\alpha)/N_\alpha.$$

By condition (a), $N_{\alpha+1}/N_\alpha$ is $\mathcal{C}$–filtered. Since

$$(\pi_K(N_{\alpha+1}) + N_\alpha)/N_\alpha \cong \pi_K(N_{\alpha+1})/\pi_K(N_\alpha),$$

$\pi_K(N_{\alpha+1})/\pi_K(N_\alpha)$ is isomorphic to a direct summand of a $\mathcal{C}$–filtered module, we infer that $\pi_K(N_{\alpha+1})/\pi_K(N_\alpha) \in \mathcal{A}$. By condition (c), $\pi_K(N_{\alpha+1})/\pi_K(N_\alpha)$ is $< \kappa$–presented. So $(\pi_K(N_{\alpha+1}) \mid \alpha \leq \tau)$ is the desired $\mathcal{A}^{<\kappa}$–filtration of $K = \pi_K(N_\tau)$. ∎

Now we arrive at the "Kaplansky Theorem for cotorsion pairs":

Theorem 2.23. *Let R be a ring, κ an uncountable regular cardinal, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a cotorsion pair of R–modules. Then the following conditions are equivalent:*

(a) *$\mathfrak{C}$ is generated by a class of $< \kappa$–presented modules;*

(b) *Every module in $\mathcal{A}$ is $\mathcal{A}^{<\kappa}$–filtered.*

Proof. (a) $\Longrightarrow$ (b). Let $\mathcal{C}$ be a class of $< \kappa$–presented modules generating $\mathfrak{C}$. W.l.o.g., $\mathcal{C}$ is a set, and $R \in \mathcal{C}$. By Corollary 1.42, $\mathcal{A}$ consists of all direct summands of $\mathcal{C}$–filtered modules. So statement (b) follows by Lemma 2.22.

(b) $\Longrightarrow$ (a). By the Eklof Lemma 1.30, every $\mathcal{A}$–filtered module is again in $\mathcal{A}$. Thus (b) implies that $\mathfrak{C}$ is generated by the class $\mathcal{A}^{<\kappa}$. ∎

The name of Theorem 2.23 above comes from the fact that its application to the cotorsion pair $(\mathcal{P}_0, \mathrm{Mod}\text{–}R)$ generated by R yields (for $\kappa = \aleph_1$) the following classical theorem on the structure of projective modules due to Kaplansky:

Corollary 2.24. *Every projective module over an arbitrary ring is a direct sum of countably generated projective modules.*

The latter application also shows that in general it is not possible to extend Theorem 2.23 to the case of $\kappa = \aleph_0$. Namely, there exist rings R which admit countably generated projective modules that are not direct sums of finitely generated projective ones.

Next, we present sufficient conditions for the completeness of a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ expressed in terms of closure properties of the classes $\mathcal{A}$ and $\mathcal{B}$. These conditions will be crucial for the classification of tilting and cotilting modules and classes in chapter 3. The proofs will again proceed via deconstruction of $\mathfrak{C}$, but in a more sophisticated way.

We will first consider an important case when each module in $\mathcal{A}$ is $\mathcal{A}^{\leq\kappa}$–filtered where $\kappa = \dim(R)$. This result comes from [71]. In order to produce filtrations with "small" consecutive factors, one has to treat filtrations of regular and singular length separately, since each of these cases requires different set–theoretic techniques. We start with the regular case.

Definition 2.25. Let κ be an infinite cardinal.

(i) For a module M, a continuous chain, $\mathcal{M} = (M_\alpha \mid \alpha \leq \kappa)$ of submodules of M is called a *κ–filtration* of M provided that $\operatorname{gen}(M_\alpha) < \kappa$ for all $\alpha < \kappa$, and $M = M_\kappa$.

(ii) A strictly ascending function $f : \kappa \to \kappa$ is called *continuous*, provided that $f(0) = 0$, and $f(\alpha) = \sup_{\beta<\alpha} f(\beta)$ for all limit ordinals $\alpha < \kappa$.

(iii) If $\mathcal{M} = (M_\alpha \mid \alpha \leq \kappa)$ is a κ–filtration of a module M, and $f : \kappa \to \kappa$ is a continuous function, then $\mathcal{M}' = (M_{f(\alpha)} \mid \alpha \leq \kappa)$ (where we put $f(\kappa) = \kappa$) is again a κ–filtration of M, called the *subfiltration* of $\mathcal{M}$ induced by f.

It is easy to see that, if κ is a regular uncountable cardinal, then any two κ–filtrations of M coincide on a closed and unbounded subset of κ, hence they possess a common subfiltration.

The following result says that, if a class of modules $\mathcal{B}$ is closed under direct sums and $M \in {}^\perp\mathcal{B}$ is equipped with a κ–filtration $\mathcal{M}$ consisting of modules from ${}^\perp\mathcal{B}$, then M is ${}^\perp\mathcal{B}$–filtered by a subfiltration of $\mathcal{M}$.

Theorem 2.26. *Let R be a ring, κ a regular uncountable cardinal, and $\mathcal{B}$ a class of modules closed under direct sums. Let $M \in {}^\perp\mathcal{B}$ be a module possessing a κ–filtration $(M_\alpha \mid \alpha \leq \kappa)$ such that $M_\alpha \in {}^\perp\mathcal{B}$ for all $\alpha < \kappa$. Then there is a strictly increasing continuous function $f : \kappa \to \kappa$ such that $M_{f(\beta)}/M_{f(\alpha)} \in {}^\perp\mathcal{B}$ for all $\alpha < \beta < \kappa$.*

Proof. Assume the claim is false. Then the set

$$E = \{\alpha < \kappa \mid \exists\beta :\ \alpha < \beta < \kappa \,\&\, M_\beta/M_\alpha \notin {}^\perp\mathcal{B}\}$$

has a non–empty intersection with each closed and unbounded subset of κ. Passing to a subfiltration, we can assume that $E = \{\alpha < \kappa \mid \operatorname{Ext}^1_R(M_{\alpha+1}/M_\alpha, \mathcal{B}) \neq 0\}$. Then for each $\alpha \in E$ there are a $B_\alpha \in \mathcal{B}$ and a homomorphism $\delta_\alpha : M_{\alpha+1}/M_\alpha \to E(B_\alpha)/B_\alpha$ that cannot be factorized through the projection $\tau_\alpha : E(B_\alpha) \to E(B_\alpha)/B_\alpha$. For $\alpha < \kappa$, $\alpha \notin E$, we put $B_\alpha = 0$ and $\delta_\alpha = 0$.

Let $I = \prod_{\alpha<\kappa} E(B_\alpha)$, $D = \bigoplus_{\alpha<\kappa} B_\alpha \,(\subseteq I)$, and $F = I/D$. For each subset $A \subseteq \kappa$, define $I_A = \{x \in I \mid x_\beta = 0 \text{ for all } \beta < \kappa, \beta \notin A\}$. In particular, $I_\kappa = I$, and $I_\alpha \cong \prod_{\beta<\alpha} E(B_\beta)$ is injective for each $\alpha \leq \kappa$.

For each $\alpha < \kappa$, we let $F_\alpha = (I_\alpha + D)/D \,(\subseteq F)$ and π_α be the epimorphism $I_\alpha \to F_\alpha$ defined by $\pi_\alpha(x) = x + D$. Then $\operatorname{Ker}(\pi_\alpha) \cong \bigoplus_{\beta<\alpha} B_\beta \,(\in \mathcal{B})$.

Let $U = \bigcup_{\alpha<\kappa} I_\alpha$. Then $D \subseteq U \subseteq I$, and we let $G = U/D \,(\subseteq F)$ and $\pi : U \to G$ be the projection.

For each $\alpha < \kappa$, define $E_\alpha = (I_{\{\alpha\}} + D)/D$. Then there is an isomorphism $\iota_\alpha : E(B_\alpha)/B_\alpha \cong E_\alpha$, and $F_{\alpha+1} = E_\alpha \oplus F_\alpha \,(\subseteq G)$. Moreover, taking $C_\alpha = (I_{(\alpha,\kappa)} + D)/D$, we have $F = F_{\alpha+1} \oplus C_\alpha$, so $G = E_\alpha \oplus F_\alpha \oplus (C_\alpha \cap G)$. Denote by ξ_α the projection onto the first component E_α in the latter decomposition of G. Then ξ_α maps $x + D \in G$ to $y + D \in E_\alpha$, where $y_\alpha = x_\alpha$ and $y_\beta = 0$ for all $\alpha \neq \beta < \kappa$.

In order to prove that $\mathrm{Ext}^1_R(M, \mathcal{B}) \neq 0$, it suffices to construct a homomorphism $\varphi : M \to G$ that cannot be factorized through π – then $\mathrm{Ext}^1_R(M, D) \neq 0$.

φ will be constructed by induction on $\alpha < \kappa$ as a union of a continuous chain of homomorphisms, $(\varphi_\alpha \mid \alpha < \kappa)$, where $\varphi_\alpha : M_\alpha \to F_\alpha$ for all $\alpha < \kappa$.

For $\alpha < \kappa$, we use the assumption of $\mathrm{Ext}^1_R(M_\alpha, \bigoplus_{\beta<\alpha} B_\beta) = 0$ to find a homomorphism $\eta_\alpha : M_\alpha \to I_\alpha$ such that $\varphi_\alpha = \pi_\alpha \eta_\alpha$. The injectivity of the module I_α yields a homomorphism $\psi_\alpha : M_{\alpha+1} \to I_\alpha$ such that $\psi_\alpha \restriction M_\alpha = \eta_\alpha$.

Denote by ρ_α the projection $M_{\alpha+1} \to M_{\alpha+1}/M_\alpha$. Define $\varphi_{\alpha+1} = \iota_\alpha \delta_\alpha \rho_\alpha + \pi_\alpha \psi_\alpha$. Then $\varphi_{\alpha+1} \restriction M_\alpha = \pi_\alpha \psi_\alpha \restriction M_\alpha = \pi_\alpha \eta_\alpha = \varphi_\alpha$.

Finally, assume there is $\phi : M \to U$ such that $\varphi = \pi\phi$. Since $U = \bigcup_{\alpha<\kappa} I_\alpha$, the set $C = \{\alpha < \kappa \mid \phi(M_\alpha) \subseteq I_\alpha\}$ is closed and unbounded in κ. So there exists $\alpha \in C \cap E$. Denote by σ the projection $I \to E(B_\alpha)$. Then ϕ induces a homomorphism $\bar{\phi} : M_{\alpha+1}/M_\alpha \to E(B_\alpha)$ defined by $\bar{\phi}\rho_\alpha(m) = \sigma(\phi(m))$ for all $m \in M_{\alpha+1}$.

By the definition of ξ_α, we have $\iota_\alpha \tau_\alpha \sigma(x) = \xi_\alpha \pi(x)$ for each $x \in U$, $\xi_\alpha \restriction F_\alpha = 0$, and $\xi_\alpha \restriction E_\alpha = \mathrm{id}$. So for each $m \in M_{\alpha+1}$, we get

$$\tau_\alpha \bar{\phi} \rho_\alpha(m) = \iota_\alpha^{-1} \xi_\alpha \pi \phi(m) = \iota_\alpha^{-1} \xi_\alpha \varphi_{\alpha+1}(m) = \iota_\alpha^{-1} \xi_\alpha \iota_\alpha \delta_\alpha \rho_\alpha(m) = \delta_\alpha \rho_\alpha(m).$$

Since ρ_α is surjective, this proves that $\tau_\alpha \bar{\phi} = \delta_\alpha$, in contradiction with the definition of δ_α. ■

The singular cardinal case will make use of a version of Shelah's Singular Compactness Theorem. For this purpose, we will need to produce a rich supply of "small" $\mathcal{Q}$–filtered submodules of M. However, Theorem 2.26 at best yields only a single chain of such submodules. The rich supply is then provided by Lemma 2.20. But first we have to define the appropriate notion of "freeness":

Definition 2.27. Let M be a module, $\mathcal{Q}$ a set of modules, and κ a regular infinite cardinal. Then M is *κ–$\mathcal{Q}$–free*, provided there is a set $\mathcal{S}_\kappa$ consisting of $< \kappa$–generated $\mathcal{Q}$–filtered submodules of M such that:

(i) $0 \in \mathcal{S}_\kappa$,

(ii) $\mathcal{S}_\kappa$ is closed under well–ordered chains of length $< \kappa$, and

(iii) each subset of M of cardinality $< \kappa$ is contained in an element of $\mathcal{S}_\kappa$.

The set $\mathcal{S}_\kappa$ is said to *witness* the κ–$\mathcal{Q}$–freeness of M. If $\mathcal{S}_\kappa$ also satisfies

(iv) M/N is $\mathcal{Q}$–filtered for each $N \in \mathcal{S}_\kappa$,

then M is called *κ–$\mathcal{Q}$–separable*, and $\mathcal{S}_\kappa$ is said to *witness* the κ–$\mathcal{Q}$–separability of M.

Clearly every κ–$\mathcal{Q}$–separable module is $\mathcal{Q}$–filtered. The following lemma says that the converse is also true under rather weak assumptions.

Lemma 2.28. *Let R be a ring, μ be an infinite cardinal and $\mathcal{Q}$ a set of $\leq \mu$–presented modules. Then M is κ–$\mathcal{Q}$–separable, whenever M is $\mathcal{Q}$–filtered and κ is a regular cardinal $> \mu$. Moreover, it is possible to choose the witnessing sets so that $\mathcal{S}_\kappa \subseteq \mathcal{S}_{\kappa'}$ for all regular cardinals such that $\mu < \kappa < \kappa'$.*

Proof. By assumption, there is a $\mathcal{Q}$–filtration, $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$, of the module M. Using the Hill Lemma 2.20, for each κ regular cardinal $> \mu$, we define $\mathcal{S}_\kappa$ as the subset of $\mathcal{F}$ consisting of all modules of the form $M(S)$ where S is a closed subset of σ of cardinality $< \kappa$. Then the inclusions $\mathcal{S}_\kappa \subseteq \mathcal{S}_{\kappa'}$ are clear, and the properties (H1), (H2), (H4), and (H3) from Theorem 2.20 imply conditions (i), (ii), (iii), and (iv) above, respectively. ∎

If $\mathcal{Q}$ consists of elements of $(\text{Mod–}R)^{<\kappa}$ for an infinite cardinal κ, then all $< \kappa$–generated $\mathcal{Q}$–filtered modules belong to $(\text{Mod–}R)^{<\kappa}$ (in particular, this is then true for all elements of $\mathcal{S}_\kappa$ defined above). This is a corollary of the following more general result on lifting filtrations of modules to filtrations of exact sequences:

Lemma 2.29. *Let R be a ring, and M be a module. Let κ be an infinite cardinal and $(M_\alpha \mid \alpha \leq \kappa)$ be a continuous chain of submodules of M such that $M = M_\kappa$. For each $\alpha < \kappa$, let*

$$\bar{\mathcal{E}}_\alpha \quad : \quad 0 \to \operatorname{Ker}(\bar\pi_\alpha) \xrightarrow{\subseteq} \bar{P}_\alpha \xrightarrow{\bar\pi_\alpha} M_{\alpha+1}/M_\alpha \to 0$$

be a short exact sequence such that $\bar{P}_\alpha$ is a projective module. Then there exists a short exact sequence

$$\mathcal{E} \quad : \quad 0 \to \operatorname{Ker}(\pi) \xrightarrow{\subseteq} P \xrightarrow{\pi} M \to 0$$

with P projective, and a continuous direct system of short exact sequences $(\mathcal{E}_\alpha \mid \alpha \leq \kappa)$ such that $\mathcal{E} = \mathcal{E}_\kappa$, and for each $\alpha \leq \kappa$, P_α is projective, $P_{\alpha+1} = P_\alpha \oplus \bar{P}_\alpha$, and the diagram

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{E}_\alpha & : & 0 & \longrightarrow & \operatorname{Ker}(\pi_\alpha) & \xrightarrow{\subseteq} & P_\alpha & \xrightarrow{\pi_\alpha} & M_\alpha & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle\subseteq} & & \downarrow{\scriptstyle\mu_\alpha} & & \downarrow{\scriptstyle\subseteq} & & \\
\mathcal{E}_{\alpha+1} & : & 0 & \longrightarrow & \operatorname{Ker}(\pi_{\alpha+1}) & \xrightarrow{\subseteq} & P_{\alpha+1} & \xrightarrow{\pi_{\alpha+1}} & M_{\alpha+1} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle\sigma_\alpha} & & \downarrow{\scriptstyle\rho_\alpha} & & \downarrow{\scriptstyle\bar\rho_\alpha} & & \\
\bar{\mathcal{E}}_\alpha & : & 0 & \longrightarrow & \operatorname{Ker}(\bar\pi_\alpha) & \xrightarrow{\subseteq} & \bar{P}_\alpha & \xrightarrow{\bar\pi_\alpha} & M_{\alpha+1}/M_\alpha & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}$$

is commutative where $\mu_\alpha : P_\alpha \to P_{\alpha+1}$ is the split inclusion of the first component, $\rho_\alpha : P_{\alpha+1} \to \bar{P}_\alpha$ the split projection on the second, $\sigma_\alpha = \rho_\alpha \restriction \operatorname{Ker}(\pi_{\alpha+1})$, and $\bar\rho_\alpha : M_{\alpha+1} \to M_{\alpha+1}/M_\alpha$ is the canonical projection.

Proof. The direct system $(\mathcal{E}_\alpha \mid \alpha \leq \kappa)$ is constructed by induction on α. First $\mathcal{E}_0 = 0 \to 0 \to 0 \to 0 \to 0$.

The non–limit step is essentially the Horseshoe Lemma from homological algebra (see e.g. [40, §8.2]): assume that the construction is completed up to some $\alpha < \kappa$. Let $P_{\alpha+1} = P_\alpha \oplus \bar{P}_\alpha$, and $\mu_\alpha : P_\alpha \hookrightarrow P_{\alpha+1}$ ($\rho_\alpha : P_{\alpha+1} \to \bar{P}_\alpha$) be the canonical inclusion (projection).

Define $\pi_{\alpha+1} : P_{\alpha+1} \to M_{\alpha+1}$ so that $\pi_{\alpha+1} \restriction P_\alpha = \pi_\alpha$ and $\bar{\pi}_\alpha = \bar{\rho}_\alpha \pi_{\alpha+1} \restriction \bar{P}_\alpha$ (this is possible, since $\bar{P}_\alpha$ is projective and $\bar{\rho}_\alpha$ is surjective). Then $\pi_{\alpha+1}$ is surjective. Since $\mathrm{Ker}(\pi_{\alpha+1}) \cap P_\alpha = \mathrm{Ker}(\pi_\alpha)$, we get commutativity of all the squares of the diagram above except the lower left one.

It is easy to check that ρ_α maps $\mathrm{Ker}(\pi_{\alpha+1})$ onto $\mathrm{Ker}(\bar{\pi}_\alpha)$, so the lower left square is also commutative, and all rows and columns of the diagram are exact sequences.

If $\alpha \leq \kappa$ is a limit ordinal, we let $\mathcal{E}_\alpha = \varinjlim_{\beta<\alpha} \mathcal{E}_\beta$. Then, by construction, $P_\alpha = \bigcup_{\beta<\alpha} P_\beta = \bigoplus_{\beta<\alpha} \bar{P}_\beta$ is projective. ∎

Of course, if $\bar{\mathcal{E}}_\alpha$ is a projective resolution of $M_{\alpha+1}/M_\alpha \in \mathcal{P}_1^{<\kappa}$ for each $\alpha < \kappa$, then $\mathcal{E}$ is a projective resolution of M ($\in \mathcal{P}_1$). If $1 \leq n < \omega$ and $M_{\alpha+1}/M_\alpha \in \mathcal{P}_n^{<\kappa}$ for each $\alpha < \kappa$, we can use the canonical decomposition of a projective resolution into a series of short exact sequences and iterate the construction of Lemma 2.29 in order to obtain the following corollary:

Corollary 2.30. *Let R be a ring, $n \geq 1$, and M be a module. Let κ be an infinite cardinal and $(M_\alpha \mid \alpha \leq \kappa)$ a κ–filtration of M. Assume that for each $\alpha < \kappa$, $M_{\alpha+1}/M_\alpha \in \mathcal{P}_n^{<\kappa}$, so there is a projective resolution $\bar{\mathcal{R}}_\alpha$ of length n of $M_{\alpha+1}/M_\alpha$ consisting of $< \kappa$–generated modules.*

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\mathcal{E}_{\alpha,i} & : & 0 \arrow[r] & S_{\alpha,i+1} \arrow[r, "\subseteq"] \arrow[d, "\subseteq"] & P_{\alpha,i} \arrow[r] \arrow[d, "\subseteq"] & S_{\alpha,i} \arrow[r] \arrow[d, "\subseteq"] & 0 \\
\mathcal{E}_{\alpha+1,i} & : & 0 \arrow[r] & S_{\alpha+1,i+1} \arrow[r, "\subseteq"] \arrow[d] & P_{\alpha+1,i} \arrow[r] \arrow[d] & S_{\alpha+1,i} \arrow[r] \arrow[d] & 0 \\
\bar{\mathcal{E}}_{\alpha,i} & : & 0 \arrow[r] & \bar{S}_{\alpha,i+1} \arrow[r, "\subseteq"] \arrow[d] & \bar{P}_{\alpha,i} \arrow[r] \arrow[d] & \bar{S}_{\alpha,i} \arrow[r] \arrow[d] & 0 \\
 & & & 0 & 0 & 0 &
\end{tikzcd}$$

Then there exists a projective resolution $\mathcal{R}$ of length n of the module M, and for each $\alpha < \kappa$, a projective resolution $\mathcal{P}_\alpha$ of length n of the module M_α consisting of $< \kappa$–generated modules such that, for all $i < n$ and $\alpha < \kappa$, the diagram above is commutative and has exact rows and columns.

Moreover, $\mathcal{E}_i = \varinjlim_{\alpha<\kappa} \mathcal{E}_{\alpha,i}$, where $\mathcal{E}_i$, $\mathcal{E}_{\alpha,i}$ and $\bar{\mathcal{E}}_{\alpha,i}$ is the i–th short exact sequence in the canonical decomposition into short exact sequences of the long exact sequence $\mathcal{R}$, $\mathcal{R}_\alpha$ and $\bar{\mathcal{R}}_\alpha$, respectively.

The version of the Singular Compactness Theorem needed here is as follows (for its proof, we refer to [35, XII.1.14 and IV.3.7]):

Lemma 2.31. *Let R be a ring, λ a singular cardinal, and $\aleph_0 \leq \mu < \lambda$. Let $\mathcal{Q}$ be a set of $\leq \mu$–presented modules, and M be a module with $gen(M) = \lambda$. Assume M is κ–$\mathcal{Q}$–free for each regular cardinal $\mu < \kappa < \lambda$. Then M is $\mathcal{Q}$–filtered.*

For a class of modules $\mathcal{C}$, and an infinite cardinal μ, denote by $\mathfrak{F}_\mu(\mathcal{C})$ the assertion: "All modules in $\mathcal{C}$ are $\mathcal{C}^{\leq\mu}$–filtered".

Lemma 2.32. *Let R be a ring and $\mathcal{B}$ a class of modules closed under direct sums. Let $\mu = \dim(R)$. Then $\mathfrak{F}_\mu({}^{\perp_\infty}\mathcal{B} \cap \mathcal{P}_n)$ implies $\mathfrak{F}_\mu({}^{\perp_\infty}\mathcal{B} \cap \mathcal{P}_{n+1})$ for each $n < \omega$.*

Proof. Assume $\mathfrak{F}_\mu({}^{\perp_\infty}\mathcal{B} \cap \mathcal{P}_n)$ holds. Let $\kappa > \mu$ be a regular uncountable cardinal. Let $M \in {}^{\perp_\infty}\mathcal{B} \cap \mathcal{P}_{n+1}$ be a λ–generated module, so there is a short exact sequence $0 \to K \hookrightarrow R^{(\lambda)} \xrightarrow{\pi} M \to 0$.

Since $M \in {}^{\perp_\infty}\mathcal{B} \cap \mathcal{P}_{n+1}$, we have $K \in {}^{\perp_\infty}\mathcal{B} \cap \mathcal{P}_n$. Let $\mathcal{Q} = {}^{\perp_\infty}\mathcal{B} \cap \mathcal{P}_n^{\leq\mu}$. By assumption and Lemma 2.28, there is a set $\mathcal{S}_\kappa$ witnessing the κ–$\mathcal{Q}$–separability of K.

Denote by $\mathcal{S}'_\kappa$ the set of all submodules $N \subseteq M$ such that there is a subset $A \subseteq \lambda$ of cardinality $< \kappa$ with $\pi(R^{(A)}) = N$ and $K \cap R^{(A)} \in \mathcal{S}_\kappa$. By Lemma 1.30 and Corollary 2.30, $\mathcal{S}_\kappa \subseteq {}^{\perp_\infty}\mathcal{B} \cap \mathcal{P}_n^{<\kappa}$, so $\mathcal{S}'_\kappa \subseteq \mathcal{P}_{n+1}^{<\kappa}$.

We claim that $\mathcal{S}'_\kappa$ witnesses the κ–$\mathcal{Q}'_\kappa$–freeness of M where $\mathcal{Q}'_\kappa = {}^{\perp_\infty}\mathcal{B} \cap \mathcal{P}_{n+1}^{<\kappa}$. Clearly $0 \in \mathcal{S}'_\kappa$, and $\mathcal{S}'_\kappa$ is closed under well–ordered unions of chains of length $< \kappa$.

Let $N = \pi(R^{(A)}) \in \mathcal{S}'_\kappa$. We have the following commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \cap R^{(A)} \arrow[r, "\subseteq"] \arrow[d, "\subseteq"] & R^{(A)} \arrow[r] \arrow[d, "\subseteq"] & N \arrow[r] \arrow[d, "\subseteq"] & 0 \\
0 \arrow[r] & K \arrow[r, "\subseteq"] \arrow[d] & R^{(\lambda)} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K/(K \cap R^{(A)}) \arrow[r] \arrow[d] & R^{(\lambda \setminus A)} \arrow[r] \arrow[d] & M/N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

Since $K \cap R^{(A)} \in \mathcal{S}_\kappa$, $K/K \cap R^{(A)}$ is $\mathcal{Q}$–filtered, it follows for all $B \in \mathcal{B}$ that $\operatorname{Ext}^i_R(K/K \cap R^{(A)}, B) = 0$. Considering the exact sequence

$$0 = \operatorname{Ext}^i_R(M, B) \to \operatorname{Ext}^i_R(N, B) \to \operatorname{Ext}^{i+1}_R(M/N, B)$$

$$\cong \operatorname{Ext}^i_R(K/K \cap R^{(A)}, B) = 0$$

for $B \in \mathcal{B}$ and $i \geq 1$, we infer that $\mathcal{S}'_\kappa \subseteq \mathcal{Q}'_\kappa$.

It remains to prove condition (iii) of Definition 2.27. Let X be a subset of M of cardinality $< \kappa$. There is a subset $A_0 \subseteq \lambda$ of cardinality $< \kappa$ such that $X \subseteq \pi(R^{(A_0)})$. Let $L_0 = K \cap R^{(A_0)}$. By Lemma 2.10, L_0 is $< \kappa$–generated, so there exists $K_0 \in \mathcal{S}_\kappa$ such that $L_0 \subseteq K_0$. Take $A_1 \supseteq A_0$ such that $K_0 \subseteq R^{(A_1)}$ and $|A_1| < \kappa$. Put $L_1 = K \cap R^{(A_1)}$. Continuing in this way, we define a sequence $K_0 \subseteq K_1 \subseteq \dots$ of elements of $\mathcal{S}_\kappa$, and a sequence $A_0 \subseteq A_1 \subseteq \dots$ of subsets of λ of cardinality $< \kappa$ such that $K \cap R^{(A_i)} \subseteq K_i$ and $K_i \subseteq R^{(A_{i+1})}$ for all $i < \omega$. Then $K' = \bigcup_{i<\omega} K_i \in \mathcal{S}_\kappa$ and $K' = K \cap R^{(A')}$, where $A' = \bigcup_{i<\omega} A_i$. So $\pi(R^{(A')})$ is an element of $\mathcal{S}'_\kappa$ containing X, and $\mathcal{S}'_\kappa$ witnesses the κ–$\mathcal{Q}'_\kappa$–freeness of M. This completes the proof of the claim.

Let $\mathcal{C} = {}^{\perp_\infty}\mathcal{B} \cap \mathcal{P}_{n+1}$. We will prove $\mathfrak{F}_\mu(\mathcal{C})$ by induction on $\lambda = \mathrm{gen}(M)$ for all $M \in \mathcal{C}$.

If $\lambda \leq \mu$, then K is $\leq \mu$-generated, and $K \in (\text{Mod-}R)^{\leq\mu}$, by Lemma 2.10. Since $K \in {}^{\perp_\infty}\mathcal{B} \cap \mathcal{P}_n$, we infer that $M \in \mathcal{C}^{\leq\mu}$.

If $\lambda > \mu$ is regular, then we select from $\mathcal{S}'_\lambda$ a λ–filtration, $\mathcal{F}$, of M. Theorem 2.26 yields a λ–subfiltration, $\mathcal{G}$, of $\mathcal{F}$ which is a ${}^{\perp}\mathcal{B}$–filtration of M. Since $0 = \mathrm{Ext}^i_R(N', B) \to \mathrm{Ext}^{i+1}_R(N/N', B) \to \mathrm{Ext}^{i+1}_R(N, B) = 0$ for all modules $N, N' \in \mathcal{F}$ with $N' \subseteq N$, $B \in \mathcal{B}$ and $i \geq 1$, we have $\mathrm{Ext}^i_R(N/N', \mathcal{B}) = 0$ for all $i \geq 2$. So $\mathcal{G}$ is actually a ${}^{\perp_\infty}\mathcal{B}$–filtration of M. By Lemma 2.11, M possesses a λ–filtration, $\mathcal{H}$, which is a $\mathcal{P}^{<\lambda}_{n+1}$–filtration of M. Let $\mathcal{J}$ be a common subfiltration of $\mathcal{G}$ and $\mathcal{H}$. Then $\mathcal{J}$ is a $\mathcal{C}^{<\lambda}$–filtration of M. By inductive hypothesis, we can refine $\mathcal{J}$ to the desired $\mathcal{C}^{\leq\mu}$–filtration of M.

If $\lambda > \mu$ is singular, then, by inductive premise, $\mathcal{S}'_\kappa$ witnesses the κ–$\mathcal{C}^{\leq\mu}$–freeness of M for each regular uncountable cardinal $\mu < \kappa < \lambda$. So the existence of a $\mathcal{C}^{\leq\mu}$–filtration of M follows by Lemma 2.31. ∎

By Theorem 2.24, any projective module over any ring is a direct sum of countably generated modules. So $\mathfrak{F}_\mu({}^{\perp_\infty}\mathcal{B} \cap \mathcal{P}_0)$ holds for any class of modules $\mathcal{B}$ and any $\mu \geq \aleph_0$. Lemma 2.32 then gives:

Theorem 2.33. *Let R be a ring, $\mu = \dim(R)$, and $\mathcal{B}$ be a class of modules closed under direct sums. Then for each $n < \omega$, all modules in ${}^{\perp_\infty}\mathcal{B} \cap \mathcal{P}_n$ are ${}^{\perp_\infty}\mathcal{B} \cap \mathcal{P}_n^{\leq\mu}$–filtered.*

Corollary 2.34. *Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair such that $\mathcal{A} \subseteq \mathcal{P}$ and $\mathcal{B}$ is closed under direct sums. Then $\mathfrak{C}$ is complete.*

Proof. If $\mathcal{A} \subseteq \mathcal{P}$, then there is $n < \omega$ with $\mathcal{A} \subseteq \mathcal{P}_n$. So Theorems 1.40 and 2.33 apply. ∎

The dual result to Corollary 2.34 also holds true, though its proof requires quite different techniques:

Theorem 2.35. *Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair such that $\mathcal{A}$ is closed under direct products. Then $\mathcal{A}$ is definable and $\mathfrak{C}$ is perfect, provided either*

(a) *$\mathcal{A}$ is closed under pure submodules, or*

(b) *$\mathfrak{C}$ is hereditary, and $\mathcal{B} \subseteq \mathcal{I}$.*

For a proof, we refer to [68] and [70].

Now, we consider a particular case, namely approximations given by the Matlis cotorsion and strongly flat modules over domains.

Definition 2.36. Let R be a domain and Q be the quotient field of R. Let M be a module.

(i) M is *Matlis cotorsion* provided that $\mathrm{Ext}^1_R(Q, M) = 0$. Denote by $\mathcal{MC}$ the class of all Matlis cotorsion modules.

(ii) M is *strongly flat* provided that $\mathrm{Ext}^1_R(M, N) = 0$ for each Matlis cotorsion module N. Denote by $\mathcal{SF}$ the class of all strongly flat modules. So $\mathcal{SF} = {}^{\perp}\mathcal{MC}$. Clearly any projective module and any divisible torsion–free module is strongly flat.

Since Q is a flat module (namely Q is the localization of R at 0), we have

$$\mathcal{I}_0 \subseteq \mathcal{RC} \subseteq \mathcal{EC} \subseteq \mathcal{MC}$$

and hence

$$\mathcal{P}_0 \subseteq \mathcal{SF} \subseteq \mathcal{FL} \subseteq \mathcal{TF}$$

for any domain R.

By Theorem 1.40, $(\mathcal{SF}, \mathcal{MC})$ is a complete cotorsion pair (generated by Q). This cotorsion pair is called the *Matlis cotorsion pair*.

Note that Mod–Q (= the class of all divisible torsion–free R–modules) is a subclass of Mod–R closed under extensions and direct limits, $(\text{Mod–}Q)^{\perp} = \mathcal{MC}$, and Q is a $\sum$–injective module. So Theorems 1.40 (a) and 1.23 give immediately

Corollary 2.37. *Let R be a domain. Then each module has an $\mathcal{MC}$–envelope, and a special $\mathcal{SF}$–precover.*

There is an explicit description of this kind of cotorsion envelope for torsion–free modules. It relies on the following result by Matlis [62] relating cotorsion modules to R–completions:

Lemma 2.38. *Let R be a domain and M be a torsion–free module. Then M is Matlis cotorsion, if and only if $M = Q^{(\kappa)} \oplus N$ where $\kappa \geq 0$ and N is reduced and R–complete.*

Proof. Since M is torsion–free, the divisible part of M equals $\bigcap_{s \in S_0} sM$, where $S_0 = R \setminus \{0\}$. Also, the divisible part of any torsion–free module is isomorphic to $Q^{(\kappa)}$ for some $\kappa \geq 0$, hence we can w. l. o .g. assume that M is (S_0–) reduced. So it remains to prove the following:

Lemma 2.39. *Assume M is reduced and torsion–free. Then M is R–complete, if and only if* $\mathrm{Ext}^1_R(Q, M) = 0$.

Proof. Assume M is not R–complete. Since M is reduced, the canonical embedding $\eta_M : M \to \widehat{M}$ yields an exact sequence $0 \to M \xrightarrow{\eta_M} \widehat{M} \to \widehat{M}/M \to 0$. Here, $\widehat{M}$ denotes the R–completion of M, so M is an RD–submodule of $\widehat{M}$, hence $\widehat{M}/M$ is torsion–free, and also $\widehat{M}/M$ is divisible. So $\widehat{M}/M \cong Q^{(\kappa)}$ for a cardinal $\kappa > 0$. However, η_M is non–split (since $\widehat{M}$ is reduced), so $\mathrm{Ext}^1_R(Q, M) \neq 0$.

Conversely, assume that M is R–complete and $\mathrm{Ext}^1_R(Q, M) \neq 0$. Then there is a non–split exact sequence $0 \to M \to N \to Q \to 0$. Consider $n \in N \setminus M$. Then for each $s \in S_0$ there is $n_s \in N$ such that $m_s = n - sn_s \in M$. Since for each $t \in T$, $m_{st} - m_s = sn_s - stn_{st} \in sN \cap M = sM$, the net $(m_s \mid s \in S)$ is

Cauchy, so has a limit $m \in M$, and $m - n = (m - m_s) - (n - m_s) \in sN$ for each $s \in S_0$. Since M is reduced and the sequence does not split, N is also reduced, so $n = m \in M$, a contradiction. ∎

Example 2.40. Let R be a domain. Then any bounded module B is Matlis cotorsion, because $\mathrm{Ext}^1_R(Q, B)$ is a bounded Q–module, so $\mathrm{Ext}^1_R(Q, B) = 0$.

Let M be a reduced torsion–free module. By Lemma 2.38, $M \in \mathcal{MC}$, iff M is R–complete. So the R–completion of M, $\widehat{M}$, is Matlis cotorsion, and $\widehat{M}/M$ is divisible and torsion–free. Hence $\widehat{M}/M$ is strongly flat. So the $\mathcal{MC}$–envelope of M is just the inclusion $\eta_M : M \hookrightarrow \widehat{M}$ (minimality follows from the fact that $\widehat{M}$ is reduced).

Similarly, the $\mathcal{MC}$–envelope of an arbitrary torsion–free module N is just the inclusion $N \hookrightarrow D \oplus \widehat{M}$ where $N = D \oplus M$ and D is the largest divisible submodule of N.

The algebraic structure of the R–completion can easily be expressed in homological terms:

Lemma 2.41. *Let R be a domain and M be a reduced torsion–free module. Then $\widehat{M} \cong \mathrm{Ext}^1_R(Q/R, M)$ and $\widehat{M}/M \cong \mathrm{Ext}^1_R(Q, M)$. Moreover, if $M = R^{(\lambda)}$ is free, then also $\widehat{M} \cong \mathrm{Hom}_R(Q/R, (Q/R)^{(\lambda)})$*

Proof. Since $\widehat{M}$ is reduced and R–complete, applying the contravariant Hom–functor $\mathrm{Hom}_R(-, \widehat{M})$ to the exact sequence $0 \to R \to Q \to Q/R \to 0$, we get from Lemma 2.39 the following exact sequence

$$0 = \mathrm{Hom}_R(Q, \widehat{M}) \to \mathrm{Hom}_R(R, \widehat{M}) \to \mathrm{Ext}^1_R(Q/R, \widehat{M}) \to \mathrm{Ext}^1_R(Q, \widehat{M}) = 0.$$

However, applying the covariant Hom–functor $\mathrm{Hom}_R(Q/R, -)$ to the exact sequence $0 \to M \to \widehat{M} \to \widehat{M}/M \to 0$, we obtain the exact sequence

$$0 = \mathrm{Hom}_R(Q/R, \widehat{M}/M) \to \mathrm{Ext}^1_R(Q/R, M) \to$$

$$\to \mathrm{Ext}^1_R(Q/R, \widehat{M}) \to \mathrm{Ext}^1_R(Q/R, \widehat{M}/M) = 0,$$

because $\widehat{M}/M \cong Q^{(\kappa)}$ for some $\kappa \geq 0$. So $\widehat{M}/M$ is torsion–free and injective.

It follows that $\widehat{M} \cong \mathrm{Hom}_R(R, \widehat{M}) \cong \mathrm{Ext}^1_R(Q/R, \widehat{M}) \cong \mathrm{Ext}^1_R(Q/R, M)$.

Similarly, since $\mathrm{Ext}^1_R(Q, \widehat{M}) = 0 = \mathrm{Hom}_R(Q, \widehat{M})$, we derive the group isomorphism $\mathrm{Ext}^1_R(Q, M) \cong \mathrm{Hom}_R(Q, \widehat{M}/M)$. Since $\mathrm{Hom}_R(Q/R, \widehat{M}/M) = 0 = \mathrm{Ext}^1_R(Q/R, \widehat{M}/M)$, also $\mathrm{Hom}_R(Q, \widehat{M}/M) \cong \mathrm{Hom}_R(Q, \widehat{M}/M) \cong \widehat{M}/M$. This shows that $\widehat{M}/M \cong \mathrm{Ext}^1_R(Q, M)$.
For $M = R^{(\lambda)}$, we also have the exact sequence

$$0 = \mathrm{Hom}_R(Q/R, Q^{(\lambda)}) \to \mathrm{Hom}_R(Q/R, (Q/R)^{(\lambda)}) \to$$

$$\to \mathrm{Ext}^1_R(Q/R, M) \to \mathrm{Ext}^1_R(Q/R, Q^{(\lambda)}) = 0,$$

proving the last assertion. ∎

The coincidence of the various classes of flat (and cotorsion) modules characterizes Prüfer and Dedekind domains. Before proving this, we recall a result of Warfield:

Lemma 2.42. *Let R be a domain. Then $\mathcal{RC} = \mathcal{MC} \cap \mathcal{I}_1$. Moreover, $\mathcal{TF} = {}^{\intercal}(\mathcal{F}_1)$, so the Warfield cotorsion pair is cogenerated by the class of all pure–injective modules of injective dimension ≤ 1.*

Proof. Let $M \in \mathcal{RC}$. Then $\operatorname{Ext}^1_R(I, M) = 0$, hence $\operatorname{Ext}^2_R(R/I, M) = 0$, for each ideal I of R. By Lemma 1.33, $M \in \mathcal{I}_1$. Conversely, let $M \in \mathcal{MC} \cap \mathcal{I}_1$ and $N \in \mathcal{TF}$. The injective hull of N is isomorphic to $Q^{(\kappa)}$ for a cardinal κ. Applying $\operatorname{Hom}_R(-, M)$ to the exact sequence $0 \to N \to Q^{(\kappa)} \to Q^{(\kappa)}/N \to 0$, we get $0 = \operatorname{Ext}^1_R(Q^{(\kappa)}, M) \to \operatorname{Ext}^1_R(N, M) \to \operatorname{Ext}^2_R(Q^{(\kappa)}/N, M) = 0$, so $M \in \mathcal{RC}$.

Since pure–injective modules are Matlis cotorsion, we have $\mathcal{TF} \subseteq {}^{\perp}(\mathcal{PI} \cap \mathcal{I}_1)$. Conversely, let $M \in {}^{\perp}(\mathcal{PI} \cap \mathcal{I}_1) = {}^{\perp}(\mathcal{D} \cap \mathcal{I}_1)$, where $\mathcal{D}$ denotes the class of all character modules of all left R–modules. Note that a module N^c has injective dimension ≤ 1, iff N has weak dimension ≤ 1, by Lemma 1.31 (b). So $M \in {}^{\intercal}\mathcal{F}_1$. Since $\{R/Rr \mid r \in R\} \subseteq \mathcal{F}_1$, we conclude that $M \in \mathcal{TF}$. ∎

Lemma 2.43. *Let R be a domain and $n > 0$. Then $\mathcal{F}_n = {}^{\perp_n}(\mathcal{DI} \cap \mathcal{PI})$. In particular, the cotorsion pair $(\mathcal{F}_1, (\mathcal{F}_1)^{\perp})$ is cogenerated by the class of all divisible pure–injective modules.*

Proof. We have $\mathcal{F}_n = {}^{\perp_{n+1}}\mathcal{PI} = {}^{\perp_n}\mathcal{C}_1$ where $\mathcal{C}_1$ is the class of all 1–st cosyzygies of all pure–injective modules. Since R is a domain, $\mathcal{C}_1 \subseteq \mathcal{DI}$. By Lemma 1.49, we get $\mathcal{F}_n \supseteq {}^{\perp_n}(\mathcal{DI} \cap \mathcal{PI})$.

Conversely, let $M \in \mathcal{F}_n$ and let N be a module such that $N^c \in \mathcal{DI}$. Then $N \in \mathcal{TF} = {}^{\intercal}(\mathcal{F}_1)$ by Lemma 2.42. So $0 = \operatorname{Tor}^R_1(\Omega^{n-1}(M), N) \cong \operatorname{Tor}^R_n(M, N)$. Since ${}^{\perp_n}(\mathcal{DI} \cap \mathcal{PI}) = {}^{\perp_n}(\mathcal{DI} \cap \mathcal{D})$ where $\mathcal{D}$ is the class of all dual modules, we conclude that $\mathcal{F}_n \subseteq {}^{\perp_n}(\mathcal{DI} \cap \mathcal{PI})$. ∎

Theorem 2.44. *Let R be a domain. The following conditions are equivalent:*

(a) *R is a Dedekind domain;*

(b) *The classes of all Warfield, Enochs, and Matlis cotorsion modules coincide;*

(c) *The classes of all torsion–free, flat, and strongly flat modules coincide.*

Proof. Clearly (b) is equivalent to (c). Also, if R is Dedekind, that is, hereditary, then Lemma 2.42 shows that $\mathcal{RC} = \mathcal{MC}$ ($= \mathcal{EC}$). So it suffices to prove that (b) implies (a):

Consider a torsion–free module F and $0 \neq r \in R$. We have the exact sequence $0 \to F \xrightarrow{\nu} F \to F/rF \to 0$ where ν is the multiplication by r. Since F/rF is bounded, we have $F/rF \in \mathcal{MC} \subseteq \mathcal{I}_1$ by assumption and by Lemma 2.42. So $\operatorname{Ext}^2_R(M, F/rF) = 0$ for any module M. So the map $\operatorname{Ext}^2_R(M, \nu) : \operatorname{Ext}^2_R(M, F) \to \operatorname{Ext}^2_R(M, F)$ (which is again multiplication by r) is surjective. It follows that $\operatorname{Ext}^2_R(M, F)$ is divisible. However, taking $M = R/I$ for an ideal I, we get that $\operatorname{Ext}^2_R(R/I, F)$ is bounded, and hence $\operatorname{Ext}^2_R(R/I, F) = 0$. This gives $\operatorname{inj\,dim} F \leq 1$.

Finally, for any module N, the torsion–free cover π of N yields an exact sequence $0 \to W \to F \xrightarrow{\pi} N \to 0$ where F is torsion–free and W is Warfield cotorsion. Since $\mathcal{I}_1$ is a coresolving class, we conclude that $\operatorname{inj\,dim} N \leq 1$. This proves that R is hereditary. ∎

Theorem 2.45. *Let R be a domain. The following conditions are equivalent:*

(a) *R is a Prüfer domain;*

(b) *$w\ gl\ dim R \leq 1$;*

(c) *The classes of all Warfield and Enochs cotorsion modules coincide;*

(d) *The classes of all torsion–free and flat modules coincide;*

(e) *All pure–injective modules have injective dimension ≤ 1.*

Proof. Assume (a). Let I be an ideal of R. Since I is a direct limit of finitely generated, and hence projective, ideals, I is flat. It follows that $\operatorname{Tor}_2^R(R/I, M) = 0$ for any module M. By the Flat Test Lemma this implies that submodules of flat modules are flat, so (b) holds true.

Assume (b). Then submodules of flat modules are flat. Since Q is flat, we see that (d) holds. Clearly (d) is equivalent to (c).

Assume (d). Since torsion–free modules are always closed under direct products, a classic result of Chase gives that R is a coherent ring. (d) also implies that each right ideal I of R is flat. If I is finitely generated, then I is finitely presented (and flat), hence projective. So (a) holds true.

Finally, for any ring R, any pure–injective module I is a direct summand in the dual module I^{cc}, so the weak global dimension of R is the supremum of injective dimensions of all pure–injective modules by Lemma 1.31 (b). In particular, (b) and (e) are equivalent for any ring R. ∎

Before turning to the remaining case of $\mathcal{FL} = \mathcal{SF}$, we need more information about the structure of the strongly flat modules. Since Q is $\sum$–injective, Corollary 1.41 immediately yields the following characterization.

Corollary 2.46. *Let R be a domain. A module M is strongly flat, if and only if there exist cardinals κ and λ and an extension $0 \to R^{(\kappa)} \to N \to Q^{(\lambda)} \to 0$ such that M is a direct summand of N.*

In particular, $\operatorname{proj\,dim} M \leq \operatorname{proj\,dim} Q$ *for any strongly flat module M. Moreover, if M is strongly flat, then $M_p = M \otimes_R R_p$ is a strongly flat R_p–module for each $p \in \operatorname{spec} R$.*

Now we can determine the structure of the kernels of the three cotorsion pairs (in the particular case of valuation domains, Corollary 2.46 and Theorem 2.47 (c) can substantially be improved — see Theorem 2.53 and Corollary 2.54 below):

Theorem 2.47.

(a) *Assume R is a right coherent ring. Then the kernel of the Enochs cotorsion pair coincides with the class of all pure–injective flat modules. If R is commutative and noetherian, then the kernel consists of all modules of the form $\prod_{p \in \operatorname{spec} R} C_p$, where C_p denotes the p-adic completion of the free module $R_{(p)}^{(\alpha_p)}$ ($\alpha_p \geq 0$) for each $0 \neq p \in \operatorname{spec} R$, and $C_p = R_{(p)}^{(\alpha)}$ ($\alpha \geq 0$) for $0 = p \in \operatorname{spec} R$, respectively.*

(b) *Assume R is a domain. Then the kernel of the Warfield cotorsion pair coincides with the class of all pure–injective torsion–free modules of injective dimension ≤ 1.*

(c) *Assume R is a domain. Then the kernel of the Matlis cotorsion pair consists of all direct summands of modules of the form $Q^{(\kappa)} \oplus \widehat{R^{(\lambda)}}$ for some cardinals κ and λ.*

Proof. (a) Let M be flat and cotorsion. Then the character module M^c is injective, and $(M^c)^c$ is flat. Since the embedding $M \hookrightarrow (M^c)^c$ is pure, also $(M^c)^c/M$ is flat. Since M is cotorsion, the sequence $0 \to M \to (M^c)^c \to (M^c)^c/M \to 0$ splits. So M is a direct summand in a dual module, hence M is pure–injective. The second statement follows from Enochs' classification of pure–injective flat modules over commutative noetherian rings, see [40, §5.3].

(b) Let $C = Q \oplus K^c$ where $K = Q/R$. Since K is divisible, K^c is torsion–free, so $\mathrm{Cogen}(C) = \mathcal{TF}$. Moreover, ${}^{\perp}C = {}^{\perp}K^c = {}^{\intercal}K = \mathcal{TF}$, so C is a 1–cotilting module. By Lemma 3.39, the kernel of the Warfield cotorsion pair coincides with $\mathrm{Prod}(C)$. Since C is torsion–free, pure–injective, and of injective dimension ≤ 1, the same is true for any module in the kernel. The reverse inclusion follows by Lemma 2.42, since $\mathcal{PI} \subseteq \mathcal{EC} \subseteq \mathcal{MC}$.

(c) Let $P \in \mathcal{SF} \cap \mathcal{MC}$. By Corollaries 1.41 and 2.46, P is a direct summand in a module N such that N is Matlis cotorsion and there is an exact sequence

$$0 \to R^{(\lambda)} \to N \to Q^{(\lambda')} \to 0 \tag{2.1}$$

for some cardinals λ and λ'. Since N is torsion–free, we have $N \cong Q^{(\kappa)} \oplus N'$ for a cardinal κ and a reduced torsion–free module N'. Since $\mathrm{Ext}_R(Q, N') = 0$, we have $N' \cong \mathrm{Hom}_R(R, N') \cong \mathrm{Ext}_R(Q/R, N')$. From (2.1), we get

$$\mathrm{Ext}_R(Q/R, R^{(\lambda)}) \cong \mathrm{Ext}_R(Q/R, N').$$

By Lemma 2.41, $\mathrm{Ext}_R(Q/R, R^{(\lambda)}) \cong \widehat{R^{(\lambda)}}$, so $N \cong Q^{(\kappa)} \oplus \widehat{R^{(\lambda)}}$.

Conversely, let $N = Q^{(\kappa)} \oplus \widehat{R^{(\lambda)}}$ for some κ and λ. Applying $\mathrm{Hom}_R(-, R^{(\lambda)})$ to the exact sequence $0 \to R \to Q \to Q/R \to 0$, we get

$$0 = \mathrm{Hom}_R(Q, R^{(\lambda)}) \to R^{(\lambda)} \to \mathrm{Ext}_R(Q/R, R^{(\lambda)}) \to \mathrm{Ext}_R(Q, R^{(\lambda)}) \to 0. \tag{2.2}$$

Since $\mathrm{Ext}_R(Q, R^{(\lambda)})$ is a Q–module, we have $\mathrm{Ext}_R(Q, R^{(\lambda)}) \cong Q^{(\kappa')}$ for a cardinal κ'. Since $\mathrm{Ext}_R(Q/R, R^{(\lambda)}) \cong \widehat{R^{(\lambda)}}$, (2.2) induces a presentation of N of the form (2.1). By Proposition 2.46, $N \in \mathcal{SF}$. By Lemma 2.38, also $N \in \mathcal{MC}$.

It follows that any direct summand of N belongs to $\mathcal{MC} \cap \mathcal{SF}$. ∎

For any domain R the classes of all flat and torsion–free modules are resolving. This may fail for the class of all strongly flat modules. The corresponding characterization goes back to Matlis:

Lemma 2.48. *Let R be a domain. Then the following conditions are equivalent:*

(a) *R is a Matlis domain;*

(b) *The class of all strongly flat modules is resolving;*

(c) *The class of all Matlis cotorsion modules is coresolving.*

Proof. We assume (a). Then $\operatorname{Ext}^n_R(Q,-)=0$ for each $n>1$ and it also follows $\operatorname{Ext}^n_R(M,-)=0$ for each $M\in\mathcal{SF}$ by Corollary 1.41. Application of Lemma 1.20, then gives (b).
(b) implies (c) by Lemma 1.20.
(c) implies (a): for a module M, denote by $h(M)$ the trace of Q in M. (So M is h–divisible, if $h(M)=M$, and h–reduced, if $h(M)=0$.)

First we prove that $\operatorname{Ext}^1_R(K,M)$ is h–reduced for each module M, where $K=Q/R$. From the exact sequence $0\to M\to E(M)\to E(M)/M\to 0$, we get the exact sequence $0\to A\to B\to \operatorname{Ext}^1_R(K,M)\to 0$, where
$A=\operatorname{Hom}_R(K,E(M))/\operatorname{Hom}_R(K,M)$ and $B=\operatorname{Hom}_R(K,E(M)/M)$. Since K is torsion, B is h–reduced. So it suffices to prove that A is Matlis cotorsion. Since K is torsion, $\operatorname{Hom}_R(K,E(M))$ is Matlis cotorsion by Lemma 1.31 (b). Since A is isomorphic to a submodule of $\operatorname{Hom}_R(K,E(M)/M)$ which is h–reduced, also A is h–reduced. It follows that $\operatorname{Hom}_R(K,M)$ is Matlis cotorsion. Now the assumption (c) gives that A is also Matlis cotorsion. This implies that $\operatorname{Ext}^1_R(K,M)$ is h–reduced.

Applying $\operatorname{Hom}_R(-,M)$ to the exact sequence $0\to R\to Q\to K\to 0$, we get the following two exact sequences:

$$0\to \operatorname{Hom}_R(K,M)\to \operatorname{Hom}_R(Q,M)\to h(M)\to 0$$

and

$$0\to M/h(M)\to \operatorname{Ext}^1_R(K,M)\to \operatorname{Ext}^1_R(Q,M)\to 0.$$

From the latter we infer that $M/h(M)$ is h–reduced for any module M.

Next consider an exact sequence $0\to T\to M\xrightarrow{\pi} Q\to 0$, where T is torsion and h–divisible. Then T is the torsion part of M. Since $T\subseteq h(M)$, there is an epimorphism $Q\cong M/T\to M/h(M)$. Since $M/h(M)$ is h–reduced, we infer that $M=h(M)$ is h–divisible.

We will prove that the torsion part of M splits, so $\operatorname{Ext}^1_R(Q,T)=0$: by assumption, there are a cardinal κ and an epimorphism $\rho: Q^{(\kappa)}\to M$. Take $m_1\in M$ and $q_1\in Q^{(\kappa)}$ such that $\pi(m_1)=1$ and $\rho(q_1)=m_1$. We will prove that $T\oplus\rho(q_1Q)=M$. First let $m\in M\setminus T$. Then $(m+T)s=(m_1+T)r$ for some non–zero $r,s\in R$, so there exists $t\in T$ such that $ms-m_1r=t$. Since T is divisible, there is $t'\in T$ with $t=t's$. So $(m-t')s=m_1r=\rho(q_1r)$, and $ms=\rho(q_1rs^{-1})s+t's$. Put $d=m-\rho(q_1rs^{-1})+t'$. Then $d\in M$ and $ds=0$, so $d\in T$, and $m\in T+\rho(q_1Q)$. However, $T\cap\rho(q_1Q)=0$, if $t=\rho(q_1rs^{-1})$ is an arbitrary element of the intersection, then $ts=\rho(q_1r)=m_1r\in T$, so $0=\pi(ts)=r$.

Finally, $\operatorname{Ext}^2_R(Q,N)\cong\operatorname{Ext}^1_R(Q,T)=0$, where $T=E(N)/N$ is torsion and h–divisible, for any module N. So $\operatorname{proj\,dim} Q\le 1$. ∎

We also note the following relation between strongly flat modules and R–completions:

Proposition 2.49. *Let R be a domain and $M\in$ Mod–R. Consider the following conditions:*

(a) *M is strongly flat.*

(b) *$M\cong Q^{(\kappa)}\oplus N$ where N is reduced torsion–free and $\widehat{N}$ is isomorphic to a direct summand of $\widehat{R^{(\lambda)}}$, for some cardinals κ and λ.*

Then (a) implies (b). If R is a Matlis domain, then (b) implies (a).

Proof. (a) implies (b): let M be strongly flat. Since M is torsion–free, $M \cong Q^{(\kappa)} \oplus N$ where N is reduced and torsion–free. Then $N \hookrightarrow \widehat{N}$ is an $\mathcal{MC}$–envelope of N. Moreover, $\widehat{N}$ is strongly flat, so $\widehat{N} \oplus X = Q^{(\rho)} \oplus \widehat{R^{(\lambda)}}$ for a module X and cardinals ρ and λ, by Theorem 2.47. Applying the functor $\mathrm{Ext}^1_R(Q/R, -)$, we get from Lemma 2.41 that $\widehat{N} \oplus \widehat{X} \cong \widehat{R^{(\lambda)}}$.

(b) implies (a): by Theorem 2.47 (c), $\widehat{N}$ is strongly flat. Moreover, there is a cardinal σ such that $0 \to N \to \widehat{N} \to Q^{(\sigma)} \to 0$ is exact. By assumption and Lemma 2.48, $\mathcal{SF}$ is resolving, whence N is strongly flat. ∎

We arrive at the remaining case of coincidence, characterized by Bazzoni and Salce in [25]. It turns out that the relevant domains here are the almost perfect ones:

Definition 2.50. Let R be a commutative ring. Then R is *almost perfect*, provided that R/I is a perfect ring for each (principal) ideal $0 \neq I \neq R$.

It is easy to see that an almost perfect domain cannot contain a strictly increasing chain of principal ideals (otherwise, if $0 \neq Rr_0 \subsetneq Rr_1 \subsetneq \dots$ is such a chain with $r_i = s_i r_{i+1}$ for all $i < \omega$, then $Rs_0 \supsetneq Rs_1 s_0 \supsetneq \dots$ is a strictly decreasing chain of principal ideals containing Rr_0, hence R/Rr_0 is not perfect). In particular, a valuation domain R is almost perfect, iff R is noetherian.

Any noetherian domain R of Krull dimension ≤ 1 is almost perfect, since then R/I is artinian for each $0 \neq I \neq R$. In fact, a coherent domain is almost perfect, iff it is noetherian of Krull dimension ≤ 1 (see [24, §4]).

Theorem 2.51. *Let R be a domain. The following conditions are equivalent:*

(a) *R is almost perfect;*

(b) *The classes of all flat and strongly flat modules coincide;*

(c) *The classes of all Matlis and Enochs cotorsion modules coincide.*

For a proof of Theorem 2.51 we refer to [24, §4].

Corollary 2.52. *Let R be a ring. Then R is a Dedekind domain, if and only if R is a Prüfer domain which is almost perfect.*

Proof. By Theorems 2.44, 2.45 and 2.51. ∎

Corollary 2.37 suggests the question of the existence of strongly flat covers of modules over domains. The (trivial) sufficient condition for the existence is $\mathcal{SF} = \mathcal{FL}$. Bazzoni and Salce proved that this condition is also necessary. In other words, all modules have strongly flat covers, iff R is almost perfect (cf. [24]).

Strongly flat modules over a domain are characterized in Corollary 2.46 as the direct summands of extensions of free modules by torsion–free divisible modules.

The drawback of this characterization is in dealing with direct summands. This is of course necessary in general: projective modules need not be extensions of free modules by torsion–free divisible modules. In fact, direct summands have to be considered, even when projective modules coincide with the free ones: for example, the group ($\mathbb{Z}$–module) of all p–adic integers J_p is (strongly) flat, but it is easily seen not to be an extension of a free group by a divisible torsion–free group.

There is a case where we do not have to consider direct summands, namely when R is a valuation domain. A proof of this fact appears in [72] and relies on an application of the rank version of the Hill Lemma 2.21:

Theorem 2.53. *Let R be a valuation domain and M be a module. Then M is strongly flat, if and only if M is an extension of a free module by a torsion–free divisible module.*

Corollary 2.54. *Let R be a valuation domain. Then the reduced strongly flat Matlis cotorsion modules coincide with the R–completions of free modules. So the kernel of the Matlis cotorsion pair consists of all modules of the form $Q^{(\kappa)} \oplus \widehat{R^{(\lambda)}}$ for some cardinals κ and λ.*

Let R be any ring and $\mathcal{C}$ be any class of modules. Recall that $\varinjlim \mathcal{C}$ denotes the class of all modules $D \in \text{Mod–}R$ such that $D = \varinjlim_{i\in I} C_i$, where $\{C_i, f_{ji} \mid i \leq j \in I\}$ is a direct system of modules from $\mathcal{C}$. Recall also that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *closed* provided that $\mathcal{A} = \varinjlim \mathcal{A}$.

The cotorsion pair $(\text{Mod–}R, \mathcal{I}_0)$ is closed, and if $(\mathcal{A}_i, \mathcal{B}_i)$ $(i \in I)$ are closed cotorsion pairs, then $(\bigcap_{i\in I} \mathcal{A}_i, (\bigcap_{i\in I} \mathcal{A}_i)^{\perp})$, is also closed. So for each cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ there is a least cotorsion pair $\overline{\mathfrak{C}} = (\overline{\mathcal{A}}, \overline{\mathcal{B}})$ such that $\overline{\mathfrak{C}}$ is closed and $\mathcal{A} \subseteq \overline{\mathcal{A}}$. $\overline{\mathfrak{C}}$ is called the *closure* of the cotorsion pair $\mathfrak{C}$.

The interesting case is when the closure of a cotorsion pair is complete – then the closure is perfect by Corollary 1.28, that is, the closure provides for envelopes and covers of modules.

Our next goal is to show that this happens when $\mathfrak{C}$ is generated by a class of FP_2–modules $\mathcal{C}$ (and in particular, when $\mathfrak{C}$ is generated by a class of finitely presented modules over a right coherent ring). As a by–product, we will describe the class $\varinjlim \mathcal{C}$ in homological terms by showing that $\varinjlim \mathcal{C} = {}^{\intercal}(\mathcal{C}^{\intercal})$. These results come from [8].

We start with several basic properties of the classes $\varinjlim \mathcal{C}$. The case when $\mathcal{C}$ consists of finitely presented modules is well-known from the classical work of Lenzing:

Lemma 2.55. *Let R be a ring and $\mathcal{C}$ be a class of finitely presented modules closed under finite direct sums. Then the following are equivalent for a module M.*

(a) $M \in \varinjlim \mathcal{C}$.

(b) *There is a pure epimorphism $f : \bigoplus_{i\in I} C_i \to M$ for a sequence $(C_i \mid i \in I)$ of modules in $\mathcal{C}$.*

(c) *Every homomorphism $h : F \to M$, where F is finitely presented, has a factorization through a module in $\mathcal{C}$.*

Moreover, $\varinjlim \mathcal{C}$ is closed under direct limits, pure submodules and pure epimorphic images, and the finitely presented modules in $\varinjlim \mathcal{C}$ are exactly the direct summands of modules in $\mathcal{C}$.

We will also need the following result by Crawley–Boevey [33] (recall that a class $\mathcal{C} \subseteq \text{Mod–}R$ is *definable*, provided that $\mathcal{C}$ is closed under direct limits, direct products and pure submodules):

Lemma 2.56. *Let R be a right coherent ring, and $\mathcal{C}$ be a class of finitely presented modules such that $\mathcal{C} = \operatorname{add}(\mathcal{C})$. Then the following are equivalent:*

(a) *$\mathcal{C}$ is covariantly finite in* mod–R*;*

(b) *$\varinjlim \mathcal{C}$ is closed under direct products;*

(c) *$\varinjlim \mathcal{C}$ is definable;*

(d) *$\varinjlim \mathcal{C}$ is a preenveloping class.*

Proof. (a) implies (b): let $(C_i \mid i \in I)$ be a sequence of modules in $\mathcal{C}$. Let $M = \prod_{i \in I} C_i$ and let $f : F \to M$ be a homomorphism with F finitely presented. Let $g : F \to C$ be a $\mathcal{C}$–preenvelope of F. Then for each $i \in I$, there is $h_i : C \to C_i$ such that $h_i g = \pi_i f$, where $\pi_i : M \to C_i$ is the canonical projection. So there is $h : C \to M$ such that $h_i = \pi_i h$ for all $i \in I$. Then $f = hg$, so $M \in \varinjlim \mathcal{C}$ by Lemma 2.55.

(b) implies (c): by Lemma 2.55, $\varinjlim \mathcal{C}$ is closed under direct limits and pure submodules, so $\varinjlim \mathcal{C}$ is definable.

(c) implies (d): let $M \in \text{Mod–}R$ and let $\kappa = |M| + |R| + \aleph_0$. Denote by $\mathcal{S}$ a representative set of all $\leq \kappa$–generated modules in $\varinjlim \mathcal{C}$. Consider the canonical map $f : M \to L$ where $L = \prod_{N \in \mathcal{S}} N^{\operatorname{Hom}_R(M,N)} \in \varinjlim \mathcal{C}$ by assumption. Then any morphism $g : M \to N$ with $N \in \mathcal{S}$ factors through f. Let $L' \in \varinjlim \mathcal{C}$ and $f' : M \to L'$. Then $|\operatorname{Im} f'| \leq \kappa$, so there is a pure submodule, N', of L' such that $\operatorname{Im} f' \subseteq N'$ and $|N'| \leq \kappa$ (cf. Lemma 1.45 (a)). By assumption, $N' \in \varinjlim \mathcal{C}$, so $N' \cong N$ for some $N \in \mathcal{S}$. It follows that f' factors through f. This proves that f is a $\varinjlim \mathcal{C}$–preenvelope of M.

(d) implies (a): let $f : M \to L$ be a $\varinjlim \mathcal{C}$–preenvelope of a module $M \in \text{mod–}R$. By Lemma 2.55, there are $C \in \mathcal{C}$, $g : M \to C$ and $h : C \to L$ such that $f = hg$. We prove that g is a $\mathcal{C}$–preenvelope of M in mod–R. Let $g' : M \to C'$ where $C' \in \mathcal{C}$. Then $g' = h'f$ for some $h' : L \to C'$, so $g' = h'hg$. ∎

If R is a right noetherian ring, then there is a simple relation between torsion pairs in Mod–R and mod–R:

Lemma 2.57. *Let R be a right noetherian ring.*

(a) *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in* Mod–R*. Then $(\mathcal{T}^{<\omega}, \mathcal{F}^{<\omega})$ is a torsion pair in* mod–R*.*

(b) *Let $(\mathcal{C}, \mathcal{D})$ be a torsion pair in* mod–R*. Then $(\varinjlim \mathcal{C}, \varinjlim \mathcal{D})$ is a torsion pair in Mod–R. Moreover, $\varinjlim \mathcal{C} = \operatorname{Gen}(\mathcal{C})$ and $\varinjlim \mathcal{D} = Ker \operatorname{Hom}_R(\mathcal{C}, -)$.*

Proof. (a) Let $M \in \text{mod–}R$ be such that $\text{Hom}_R(M, \mathcal{F}^{<\omega}) = 0$. Since each $N \in \mathcal{F}$ is a directed union of its finitely generated submodules (in $\mathcal{F}^{<\omega}$) and R is right noetherian, we have $\text{Hom}_R(M, \mathcal{F}) = 0$ by Lemma 1.34, so $M \in \mathcal{T}^{<\omega}$.

Conversely, let $N \in \text{mod–}R$ be such that $\text{Hom}_R(\mathcal{T}^{<\omega}, N) = 0$. Assume there is $0 \neq f \in \text{Hom}_R(T, N)$ where $T \in \mathcal{T}$. Then $\text{Im} f \in \mathcal{T}^{<\omega}$, since R is right noetherian, a contradiction. So $\text{Hom}_R(\mathcal{T}, N) = 0$, and $N \in \mathcal{F}^{<\omega}$.
Since $\text{Hom}_R(\mathcal{T}^{<\omega}, \mathcal{F}^{<\omega}) = 0$, the claim follows.
Next we show (b): by Lemma 1.34, $\text{Hom}_R(\mathcal{C}, \varinjlim \mathcal{D}) = 0$. If $M \in \varinjlim \mathcal{C}$, then $\text{Hom}_R(M, \varinjlim \mathcal{D}) = 0$ by Lemma 2.55. So $\text{Hom}_R(\varinjlim \mathcal{C}, \varinjlim \mathcal{D}) = 0$.

If $M \in \text{Ker}\,\text{Hom}_R(\mathcal{C}, -)$, then each finitely presented submodule of M belongs to $\mathcal{D}$, hence $M \in \varinjlim \mathcal{D}$. It follows that

$$\varinjlim \mathcal{D} = \text{Ker}\,\text{Hom}_R(\mathcal{C}, -) = \text{Ker}\,\text{Hom}_R(\varinjlim \mathcal{C}, -).$$

Let M be any module and let $\mathcal{S}$ be the direct system of all its finitely presented submodules. For $F \in \mathcal{S}$, denote by $t(F) \in \mathcal{C}$ the torsion part of F. There is an exact sequence $0 \to t(F) \to F \to F/t(F) \to 0$ with $F/t(F) \in \mathcal{D}$. Since $t(F) \subseteq t(G)$ for all $F \subseteq G \in \mathcal{S}$, we get the induced direct system of exact sequences whose direct limit is $0 \to \varinjlim t(F) \to M \to \varinjlim F/t(F) \to 0$. In particular, if $M \in \text{Ker}\,\text{Hom}_R(-, \varinjlim \mathcal{D})$, then $M \cong \varinjlim t(F) \in \varinjlim \mathcal{C}$.

Finally, being a torsion class containing $\mathcal{C}$, $\varinjlim \mathcal{C}$ contains $\text{Gen}(\mathcal{C})$. The reverse inclusion follows from the fact that any direct limit is canonically a (pure–) epimorphic image of a direct sum. ■

Lemma 2.57 will be essential for characterizing 1–cotilting classes over noetherian rings in chapter 3.

Definition 2.58. Let $\mathcal{C}$ be a class of modules. We will denote by $\widetilde{\mathcal{C}}$ the class of all pure–epimorphic images of elements of $\mathcal{C}$. Note that $\mathcal{C} \cap \text{mod–}R = \widetilde{\mathcal{C}} \cap \text{mod–}R$, if $\mathcal{C}$ is closed under direct summands. For example, if $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair, then the class $\widetilde{\mathcal{A}}$ coincides with the class of all modules M such that each (or some) special $\mathcal{A}$–precover of M is a pure epimorphism.

Note that $\varinjlim \mathcal{C} \subseteq {}^{\intercal}(\mathcal{C}^{\intercal})$, and $\widetilde{\mathcal{C}} \subseteq {}^{\intercal}(\mathcal{C}^{\intercal})$, since ${}^{\intercal}(\mathcal{C}^{\intercal})$ is obviously closed under direct limits and pure–epimorphic images. Moreover, we have

Lemma 2.59. *Let R be a ring, $\mathcal{C}$ be a class of modules, and $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair generated by $\mathcal{C}$. Denote by $\mathcal{D}$ the class of all dual modules (= character modules of left R–modules). Then*

(a) ${}^{\intercal}(\mathcal{C}^{\intercal}) = {}^{\perp}(\mathcal{B} \cap \mathcal{D}) = {}^{\intercal}(\mathcal{A}^{\intercal})$.

(b) *Assume $\mathcal{C}$ is closed under direct sums. Then $\varinjlim \mathcal{C} \subseteq \widetilde{\mathcal{C}} \subseteq {}^{\intercal}(\mathcal{C}^{\intercal})$.*

(c) *Assume that $\mathcal{C}$ consists of FP_2–modules. Then $M \in \mathcal{B}$, iff $M^{dd} \in \mathcal{B}$ for any module M. In particular, ${}^{\intercal}(\mathcal{C}^{\intercal}) = {}^{\perp}(\mathcal{B} \cap \mathcal{PI})$.*

Proof. (a) Let M be a module. By Lemma 1.31 (b), $M \in {}^{\intercal}(\mathcal{C}^{\intercal})$, iff $M \in {}^{\perp}N^d$ for all $N \in \mathcal{C}^{\intercal}$. Moreover, $N \in \mathcal{C}^{\intercal}$, iff $N^d \in \mathcal{C}^{\perp} \cap \mathcal{D} = \mathcal{B} \cap \mathcal{D}$. For $\mathcal{C} = \mathcal{A}$, we get in particular that ${}^{\intercal}(\mathcal{A}^{\intercal}) = {}^{\perp}(\mathcal{B} \cap \mathcal{D})$.

(b) This is clear, since the assumption implies that $\widetilde{\mathcal{C}}$ is closed under direct limits.

(c) Let M be a module. By Lemma 1.31 (d), $M \in \mathcal{B}$, iff $M^d \in \mathcal{C}^\intercal$, iff $M^{dd} \in \mathcal{B}$.

Since each pure–injective module M is a direct summand in M^{dd}, we have ${}^\perp(\mathcal{B} \cap \mathcal{D}) = {}^\perp(\mathcal{B} \cap \mathcal{PI})$, and the assertion follows from part (a). ∎

Lemma 2.60. *Let R be a ring. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair such that $\mathcal{B}$ is closed under ^{dd}. Then $\widetilde{\mathcal{A}} = {}^\intercal(\mathcal{A}^\intercal)$.*

Proof. Let $M \in {}^\intercal(\mathcal{C}^\intercal)$. By Lemma 2.59 (b), $M \in {}^\perp(\mathcal{B} \cap \mathcal{D})$. Let $0 \to B \xrightarrow{\mu} A \to M \to 0$ be an exact sequence with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Consider the canonical pure embedding $\nu : B \to B^{dd}$ and take the pushout of μ and ν:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\mu} & A & \longrightarrow & M & \longrightarrow & 0 \\ & & {\scriptstyle\nu}\big\downarrow & & {\scriptstyle\eta}\big\downarrow & & \big\| & & \\ 0 & \longrightarrow & B^{dd} & \xrightarrow{\tau} & N & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

By assumption, $B^{dd} \in \mathcal{B} \cap \mathcal{D}$, so the bottom row splits. It follows that ν factors through μ, hence μ is a pure monomorphism, and $M \in \widetilde{\mathcal{A}}$. ∎

By Lemma 2.59, each cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is contained in the complete and closed cotorsion pair cogenerated by the class $\mathcal{B} \cap \mathcal{D}$. We will now investigate whether the latter is the closure of $\mathfrak{C}$. As a by–product, we obtain a homological description of the class $\varinjlim \mathcal{C}$ in the case when $\mathcal{C}$ consists of FP$_2$–modules:

Theorem 2.61. *Let R be a ring. Let $\mathcal{C}$ be a class consisting of FP$_2$–modules such that $\mathcal{C}$ is closed under extensions, direct summands and $R \in \mathcal{C}$. Then $\varinjlim \mathcal{C} = {}^\intercal(\mathcal{C}^\intercal)$ is a covering class.*
Furthermore, if $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is the cotorsion pair generated by $\mathcal{C}$, then $\varinjlim \mathcal{C} = \varinjlim \mathcal{A} = \widetilde{\mathcal{A}} = {}^\intercal(\mathcal{A}^\intercal)$.

Proof. By Lemmas 2.59 (b) and 2.60, $\widetilde{\mathcal{A}} = {}^\intercal(\mathcal{A}^\intercal) = {}^\intercal(\mathcal{C}^\intercal)$.

Now we show that $\mathcal{A} \subseteq \varinjlim \mathcal{C}$. First the isomorphism classes of $\mathcal{C}$ form a set, so $\mathcal{A}$ consists of all direct summands of $\mathcal{C}$–filtered modules by Corollary 1.42. By Lemma 2.55, $\varinjlim \mathcal{C}$ is closed under direct limits, hence under direct summands. So it suffices to prove that $\varinjlim \mathcal{C}$ contains all $\mathcal{C}$–filtered modules.

We proceed by induction on the length δ of the filtration. The cases when $\delta = 0$ and δ is a limit ordinal are clear (the latter by Lemma 2.55). Let δ be non–limit, so we have an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ with $A \in \varinjlim \mathcal{C}$ and $C \in \mathcal{C}$. We will apply Lemma 2.55 to prove that $B \in \varinjlim \mathcal{C}$.

Let $h : F \to B$ be a homomorphism with F finitely presented. Since C is FP$_2$, there is a presentation $0 \to G \to P \xrightarrow{p} C \to 0$ with P finitely generated projective and G finitely presented. There is also $q : P \to B$ such that $p = gq$. We have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \xrightarrow{f'} & F \oplus P & \xrightarrow{(gh)\oplus p} & C & \longrightarrow & 0 \\ & & {\scriptstyle h'}\big\downarrow & & {\scriptstyle h\oplus q}\big\downarrow & & \big\| & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0. \end{array}$$

Considering the pullback of p and $(gh) \oplus p$, we see that the pullback module U is an extension of G by $F \oplus P$, and F' is isomorphic to a direct summand in U. So U, and F', are finitely presented. Since $A \in \varinjlim \mathcal{C}$, Lemma 2.55 provides for a module $C' \in \mathcal{C}$ and maps $\sigma' : F' \to C'$, $\tau' : C' \to A$ such that $h' = \tau'\sigma'$. Consider the pushout of f' and σ':

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F' & \xrightarrow{f'} & F \oplus P & \xrightarrow{gh \oplus p} & C & \longrightarrow & 0 \\
 & & {\scriptstyle\sigma'}\downarrow & & {\scriptstyle\sigma}\downarrow & & \| & & \\
0 & \longrightarrow & C' & \xrightarrow{\rho} & D & \longrightarrow & C & \longrightarrow & 0.
\end{array}$$

By assumption, $D \in \mathcal{C}$. By the pushout property, there is $\tau : D \to B$ such that $\tau\sigma = h \oplus q$, hence $\tau\sigma \restriction F = h$. So h factors through D, and $B \in \varinjlim \mathcal{C}$.

Now, since $\varinjlim \mathcal{C}$ is closed under pure–epimorphic images by Lemma 2.55, we infer that $\widetilde{\mathcal{A}} \subseteq \varinjlim \mathcal{C}$. So, by Lemma 2.59, $\varinjlim \mathcal{C} = \varinjlim \mathcal{A} = \widetilde{\mathcal{A}}$.

Finally, $\varinjlim \mathcal{C}$ is a covering class by Theorem 1.48. ∎

Note that in the setting of Theorem 2.61, the class $\mathcal{A} = {}^{\perp}(\mathcal{C}^{\perp})$ consists of all direct summands of $\mathcal{C}$–filtered modules, while $\varinjlim \mathcal{A} = \varinjlim \mathcal{C} = {}^{\intercal}(\mathcal{C}^{\intercal})$ of all pure–epimorphic images of $\mathcal{C}$–filtered modules. Both classes contain the same finitely presented modules, namely the modules in $\mathcal{C}$.

Theorem 2.61 may fail, if $\mathcal{C}$ does not consist of FP_2–modules. For example, if R is a domain and Q is its quotient field, then $\mathcal{C} = \text{Mod–}Q$ is a subclass of Mod–R closed under direct limits, but $\mathcal{A} = {}^{\perp}(\mathcal{C}^{\perp}) = \mathcal{SF}$, the class of all strongly flat modules, while ${}^{\intercal}(\mathcal{C}^{\intercal}) = \mathcal{FL}$, the class of all flat modules. So, if R is not almost perfect, then $\mathcal{C} = \varinjlim \mathcal{C} \subsetneq \mathcal{A} \subsetneq {}^{\intercal}(\mathcal{C}^{\intercal}) = {}^{\intercal}(\mathcal{A}^{\intercal})$.

Corollary 2.62. *Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair generated by a class of FP_2–modules. (For example, let R be right coherent and $\mathfrak{C}$ be generated by a class of finitely presented modules.) Then the closure $\overline{\mathfrak{C}} = (\overline{\mathcal{A}}, \overline{\mathcal{B}})$ of $\mathfrak{C}$ is cogenerated by the class $\mathcal{B} \cap \mathcal{PI}$. In particular, $\overline{\mathfrak{C}}$ is perfect, and $\overline{\mathcal{A}} = \varinjlim \mathcal{A} = \widetilde{\mathcal{A}} = {}^{\intercal}(\mathcal{A}^{\intercal})$.*

Proof. If $\mathfrak{C}$ is generated by a class of FP_2–modules $\mathcal{D}$, we let $\mathcal{C}$ be the smallest class of modules closed under extensions and containing $\mathcal{D} \cup \{R\}$. Then $\mathcal{C}$ also consists of FP_2–modules, and it generates $\mathfrak{C}$. So Theorem 2.61 gives $\varinjlim \mathcal{A} = {}^{\perp}(\mathcal{B} \cap \mathcal{PI})$ and $\overline{\mathfrak{C}} = (\varinjlim \mathcal{A}, (\varinjlim \mathcal{A})^{\perp})$. Finally, $\overline{\mathfrak{C}}$ is perfect by Theorem 1.48. ∎

We finish by a result that will be essential for a characterization of 1–cotilting classes of cofinite type in chapter 3:

Lemma 2.63. *Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair such that $\mathcal{A} \subseteq \mathcal{P}_1$. Then $\mathcal{A} \subseteq \varinjlim \mathcal{A}^{<\omega}$.*

Proof. Let $A \in \mathcal{A}$. By assumption and by Eilenberg's trick, there is an exact sequence $0 \to F \subseteq G \to A \to 0$ where F and G are free modules. Let $\{x_\alpha \mid \alpha < \kappa\}$ and $\{y_\beta \mid \beta < \lambda\}$ be free bases of F and G, respectively. W.l.o.g., κ is infinite. For each finite subset $S \subseteq \kappa$, let S' be the least (finite) subset of λ such that $F_S = \bigoplus_{\alpha \in S} x_\alpha R \subseteq G_S = \bigoplus_{\beta \in S'} y_\beta R$. Then F is a directed union of its summands of the form F_S, where S runs over all finite subsets of

κ. Let $A_S = G_S/F_S$. Then $A_S \in \mathcal{P}_1^{<\omega}$, and $A = P \oplus H$, where P is free and $H = \varinjlim_S A_S$. So it remains to prove that $H \in \varinjlim \mathcal{A}^{<\omega}$.

We will show that $A_S \in \mathcal{A}^{<\omega}$ for each finite subset $S \subseteq \kappa$. Take an arbitrary $B \in \mathcal{B}$. Then any homomorphism from F to B extends to G. Let φ be a homomorphism from F_S to B. Since F_S is a direct summand in F, φ extends to F, hence to G, and G_S. It follows that $B \in \{A_S\}^\perp$, so $A_S \in \mathcal{A}^{<\omega}$, and $H \in \varinjlim \mathcal{A}^{<\omega}$. ∎

In general, Lemma 2.63 fails for $n > 1$, even for $\mathcal{A} = \mathcal{P}_n$. Smalø constructed for each $n > 1$ a finite dimensional algebra R_n such that $\varinjlim \mathcal{P}_n^{<\omega} = \mathcal{P}_1 \subsetneq \mathcal{P}_n$ (see [69]).

3 Tilting and cotilting modules

We start with a definition of a (infinitely generated) tilting module:

Definition 3.1. Let R be a ring. A module T is *tilting* provided that

(T1) T has finite projective dimension (that is, $T \in \mathcal{P}$),

(T2) $\mathrm{Ext}_R^i(T, T^{(\kappa)}) = 0$ for all $1 \leq i < \omega$ and all cardinals κ, and

(T3) There are $r \geq 0$ and a long exact sequence $0 \to R \to T_0 \to \ldots \to T_r \to 0$, where $T_i \in \mathrm{Add}(T)$ for all $i \leq r$.

If $n < \omega$ and T is tilting of projective dimension $\leq n$, then T is called *n–tilting.* The class $T^{\perp_\infty}$ is called the *n–tilting class* induced by T. Clearly $({}^\perp(T^{\perp_\infty}), T^{\perp_\infty})$ is a hereditary cotorsion pair, called the *n–tilting cotorsion pair* induced by T. If T and T' are tilting modules, then T is said to be *equivalent* to T' provided that the induced tilting classes coincide, that is, $T^{\perp_\infty} = (T')^{\perp_\infty}$.
A ring S is called *tilted* from R, if there is a tilting module T such that $S \cong \mathrm{End}(T_R)$.

A tilting module T is *classical tilting*, iff T satisfies the following stronger version of condition (T1):
(T1′) $T \in \mathcal{P}^{<\omega}$.

In view of Lemma 1.34, classical tilting modules can equivalently be defined as the modules T satisfying conditions (T1′), (T2′) and (T3′), where (T2′) says that $\mathrm{Ext}_R^i(T, T) = 0$ for all $1 \leq i < \omega$, and (T3′) is obtained from (T3) by replacing $\mathrm{Add}(T)$ with $\mathrm{add}(T)$.

Classical tilting modules are of particular importance. The fundamental result of classical tilting theory – the Tilting Theorem – says that for any classical tilting module T and any $i \leq n = \mathrm{proj\,dim}\, T$ there is a *tilting category equivalence* between the categories $\bigcap_{j \neq i} \mathrm{Ker}\, \mathrm{Ext}_R^j(T, -)$ and $\bigcap_{j \neq i} \mathrm{Ker}\, \mathrm{Tor}_j^S(-, T)$ (where $S = \mathrm{End}(T_R)$):

$$\bigcap_{j \neq i} \mathrm{Ker}\, \mathrm{Ext}_R^j(T, -) \underset{\mathrm{Tor}_i^S(-,T)}{\overset{\mathrm{Ext}_R^i(T,-)}{\rightleftarrows}} \bigcap_{j \neq i} \mathrm{Ker}\, \mathrm{Tor}_j^S(-, T)\,. \tag{3.1}$$

In fact, if T is a classical n–tilting right R–module, then T is necessarily also a classical tilting left S–module, and the bimodule ${}_S T_R$ is faithfully balanced (so T is a *tilting bimodule*) – see [65].

Rather than studying equivalences induced by classical tilting modules, we will concentrate here on relations between tilting theory of (infinitely generated) tilting modules over arbitrary rings on the one hand, and the approximation theory on the other.

Clearly 0–tilting modules coincide with (possibly infinitely generated) projective generators.

We will now present several examples of infinitely generated 1–tilting modules that naturally occur in various parts of module theory.

Example 3.2. Let R be a domain, and S a multiplicative subset of R. Let $\delta_S = F/G$, where F is the free module with the basis given by all sequences $(s_0, \dots, s_n)$ where $n \geq 0$, and $s_i \in S$ for all $i \leq n$, and the empty sequence $w = ()$; the submodule G is generated by the elements of the form $(s_0, \dots, s_n)s_n - (s_0, \dots, s_{n-1})$, where $0 < n$ and $s_i \in S$ for all $i \leq n$, and of the form $(s)s - w$, where $s \in S$.

It is easy to see that G is free, so δ_S has projective dimension ≤ 1.
By definition, δ_S is S*–divisible* (that is, for each $s \in S$, $\delta_S s = \delta_S$, or equivalently, $\mathrm{Ext}^1_R(R/sR, \delta_S) = 0$). Moreover, $\delta_S = \bigcup_{n<\omega} \delta_i$, where $\delta_0 = wR \cong R$ and δ_{i+1}/δ_i is isomorphic to the direct sum of copies of the cyclically presented modules R/sR with $s \in S$, for each $i < \omega$. From Lemma 1.30 follows $\mathrm{Ext}^1_R(\delta_S, \delta_S^{(\kappa)}) = 0$ for any cardinal κ.

For $x \in F$, denote by $\bar{x} \in \delta_S$ the coset of x in F/G. Define $\mu : \delta_S \to \delta_S$ by $\mu(\bar{w}) = 0$, and $\mu(\bar{x}) = \overline{(1, x)}$ for $\bar{x} \neq \bar{w}$. Then $\mathrm{Ker}(\mu) = wR$, so μ induces an embedding $\nu : \delta_S/wR \to \delta_S$. Conversely, define $\pi : \delta_S \to \delta_S/wR$ by $\pi(\overline{(s_0, \dots, s_n)}) = \overline{(s_1, \dots, s_n)} + wR$ for $0 < n$ and $s_i \in S$ $(i \leq n)$, and $\pi(\overline{(s)}) = \pi(\overline{w}) = 0$ for $s \in S$. Then $\pi\nu = \mathrm{id}$, so δ_S/wR is a direct summand of δ_S. It follows that δ_S is a 1–tilting module.

Note that δ_S is a generator for the class of all S–divisible modules: indeed, if M is S–divisible and $a \in M$, then there is a homomorphism $\eta : \delta_S \to M$ such that $a \in \mathrm{Im}(\eta)$. The homomorphism η is constructed by induction: first we define η on $\delta_0 \cong R$, so that its image contains a. If η is already defined on δ_i for some $i < \omega$, we use the fact that δ_{i+1}/δ_i is isomorphic to the direct sum of copies of R/sR with $s \in S$ to infer that $\mathrm{Ext}^1_R(\delta_{i+1}/\delta_i, M) = 0$, and to extend η from δ_i to δ_{i+1}.

The module $\delta = \delta_{R\setminus\{0\}}$ was introduced by Fuchs. Facchini [41] proved that δ is a 1-tilting module. The general case of δ_S comes from [43], so we will call δ_S the *Fuchs tilting module.* Notice that the 1–tilting class induced by δ_S coincides with the class of all S–divisible modules.

If R is a Prüfer domain, then any module of projective dimension ≤ 1 is filtered by finitely presented cyclic modules and $\mathrm{Ext}^1_R(R/I, D) = 0$ for any finitely generated ideal I and any $D \in \mathcal{DI}$ (see [44, §I.7 and §VI.6]). If R is a Matlis domain, then $\mathcal{DI} = \mathrm{Gen}(Q)$, and $\mathcal{P}_1 = {}^\perp \mathrm{Gen}(Q)$, [44, §VIII]. So in either case, the 1–tilting cotorsion pair induced by δ is $(\mathcal{P}_1, \mathcal{DI})$.

Example 3.3. Let R be a commutative 1–Iwanaga–Gorenstein ring (that is, a commutative noetherian ring with $\operatorname{inj\,dim} R \leq 1$, see Example 2.14). Let P_0 and P_1 denote the sets of all prime ideals of height 0 and 1, respectively. By a classical result of Bass [64, §18], the minimal injective coresolution of R has the form

$$0 \to R \to \bigoplus_{q \in P_0} E(R/q) \xrightarrow{\pi} \bigoplus_{p \in P_1} E(R/p) \to 0.$$

Consider a subset $P \subseteq P_1$. Put $R_P = \pi^{-1}(\bigoplus_{p \in P} E(R/p))$ and $T_P = R_P \oplus \bigoplus_{p \in P} E(R/p)$. We will show that T_P is a 1-tilting module (it is called the *Bass tilting module*).

First we have $R_P/R \cong \bigoplus_{p \in P} E(R/p)$ and $Q/R_P \cong \bigoplus_{p \in P_1 \setminus P} E(R/p)$. Since both R_P and R_P/R have injective (equivalently, projective) dimension ≤ 1, so does T_P. As $\operatorname{Hom}_R(E(R/p), Q/R_P) = 0$, we see that $\operatorname{Ext}^1_R(E(R/p), R_P) = 0$ for all $p \in P$. It follows that $\operatorname{Ext}^1_R(T_P, T_P^{(\kappa)}) = 0$ for each cardinal κ. Finally, the exact sequence $0 \to R \to R_P \to \bigoplus_{p \in P} E(R/p) \to 0$ yields condition (T3) for T_P.

Notice that the 1–tilting class induced by T_P is $\{M \mid \operatorname{Ext}^1_R(E(R/p), M) = 0 \text{ for all } p \in P\}$. This class equals $\{M \mid \operatorname{Ext}^1_R(R/p, M) = 0 \text{ for all } p \in P\}$ in case R is hereditary (in particular, when R is a Dedekind domain).

Example 3.4. Let R be a connected tame hereditary algebra over a field k. Let G denote the generic module. Then $S = \operatorname{End}(G)$ is a skew–field and $\dim_S Q = n < \omega$. Denote by $\mathcal{T}$ the set of all tubes. If $\alpha \in \mathcal{T}$ is a homogeneous tube, we denote by R_α the corresponding Prüfer module. If $\alpha \in \mathcal{T}$ is not homogenous, denote by R_α the direct sum of all Prüfer modules corresponding to the rays in α. There is an exact sequence

$$0 \to R \to Q^{(n)} \xrightarrow{\pi} \bigoplus_{\alpha \in \mathcal{T}} R_\alpha^{(\lambda_\alpha)} \to 0,$$

where $\lambda_\alpha > 0$ for all $\alpha \in \mathcal{T}$.

Let $P \subseteq \mathcal{T}$. Put $R_P = \pi^{-1}(\bigoplus_{\alpha \in P} R_\alpha^{(\lambda_\alpha)})$. Similarly, as in Example 3.3, we see that $T_P = R_P \oplus \bigoplus_{\alpha \in P} R_\alpha$ is a 1-tilting module, called the *Ringel tilting module*. Notice that the 1–tilting class induced by T_P equals $T_P^\perp = \{M \in \text{Mod-}R \mid \operatorname{Ext}^1_R(N, M) = 0 \text{ for all (simple) regular modules } N \in P\}$. In particular, if $P \neq P' \subseteq \mathcal{T}$, then the tilting modules T_P and $T_{P'}$ are not equivalent.

The modules in the tilting class $\mathcal{R} = T_{\mathcal{T}}^\perp = \{D \in \text{Mod-}R \mid \operatorname{Ext}^1_R(M, D) = 0 \text{ for all (simple) regular modules } M\}$ are called *Ringel divisible* (see [34]).

Example 3.5. The following examples go back to [57] (and originate in the work of Lukas [61]).

(a) Let R be a connected wild hereditary algebra over a field k. Denote by τ the Auslander–Reiten translation, and by $\mathcal{R}$ the class of all Ringel divisible modules (they are defined as in the tame case, by $\operatorname{Ext}^1_R(M, D) = 0$ for each regular module M).

Let $M, N \in \text{mod–}R$. Assume M is regular. By [61], there is an exact sequence $0 \to N \to A_M \to B_M \to 0$, where $A_M \in M^\perp$ and B_M is a finite direct sum of copies of $\tau^n M$ for some $n < \omega$.

Let X be any regular module and $\mathcal{T}_X = \{\tau^m X \mid m < \omega\}$. Iterating the construction above (for $M = X$ and $N = R$, $M = \tau X$ and $N = A_X$, etc.), we obtain an exact sequence $0 \to R \to C_X \to D_X \to 0$, where D_X has a countable $\mathcal{T}_X$–filtration. Then $T_X = C_X \oplus D_X$ is a 1–tilting module, called the *Lukas divisible module*. The corresponding 1–tilting class is $\mathcal{R}$ (so in contrast to Example 3.4, T_X and $T_{X'}$ are equivalent for all regular modules X and X').

(b) Let R be a connected hereditary algebra of infinite representation type over a field k. Let $P \neq 0$ be any preprojective module. Then there is a chain of preprojective modules A_n ($n < \omega$) with the following properties: $A_0 = R$, and for each $n < \omega$ there is an exact sequence $0 \to A_n \subseteq A_{n+1} \to P_n \to 0$, where A_{n+1} and P_n are preprojective and $\mathrm{Hom}_R(A_{n+1}, \tau^{-n}P) = 0$. Put $A_P = \bigcup_{n<\omega} A_n$, $B_P = A_P/A$, and $T_P = A_P \oplus B_P$. Then T_P is a tilting module (called the *Lukas tilting module*).

T_P induces the tilting class $T_P^{\perp} = \mathcal{L} = \{M \in \text{Mod-}R \mid \mathrm{Ext}^1_R(Q, M) = 0$ for all preprojective modules $Q \in \text{mod–}R\}$ ($\mathcal{L}$ is called the class of all $\mathcal{P}^\infty$*-torsion modules*). Again, this class does not depend on the choice of P, that is, all the Lukas tilting modules T_P are equivalent.

If R is tame, then the tilting cotorsion pair induced by T_P is $(\mathcal{B}, \mathcal{L})$ where $\mathcal{B} = {}^{\perp}\,\mathrm{Gen}(t)$, and $t \subseteq \text{mod–}R$ is the class of all regular modules. The modules $B \in \mathcal{B}$ are called *Baer modules*; for their structure, we refer to [9].

Now we consider an example of an infinitely generated n–tilting module:

Example 3.6. Let $n \geq 0$ and R be an n–Iwanaga–Gorenstein ring (see Example 2.14). Let $0 \to R \to I_0 \to \ldots \to I_n \to 0$ be the minimal injective coresolution of R.

Then $T = \bigoplus_{i \leq n} I_i$ is easily seen to be an n–tilting module: indeed, since T is injective, T has projective dimension $\leq n$, so condition (T1) of Definition 3.1 is satisfied. Since R is noetherian, $T^{(\kappa)}$ is also injective, so (T2) holds. The minimal injective coresolution above yields condition (T3). (This tilting module will play a crucial role in proving the first Bass' finitistic dimension conjecture for R in Chapter 5.)

Theorem 3.25 below will provide a characterization of all n–tilting classes. For its proof, we will need a number of basic facts on tilting modules and tilting cotorsion pairs. We start with a definition:

Definition 3.7. Let R be a ring, $\mathcal{C}$ be a class of modules, and $M \in \text{Mod–}R$.

(i) M is called $\mathcal{C}$*–resolved*, if there is a $\mathcal{C}$*–resolution of* M, that is a long exact sequence $\ldots \to C_n \to \ldots \to C_0 \to M \to 0$ such that $C_n \in \mathcal{C}$ for all $n < \omega$.

Assume M is $\mathcal{C}$–resolved. If M has a $\mathcal{C}$–resolution such that $C_i = 0$ for all $i \geq n+1$, then the least such n (among all such $\mathcal{C}$–resolutions) is called the $\mathcal{C}$*–resolution dimension* of M. Otherwise M is said to have $\mathcal{C}$–resolution dimension ∞.

(ii) Dually, M is called $\mathcal{C}$*–coresolved*, if there is a $\mathcal{C}$*–coresolution of* M, that is a long exact sequence $0 \to M \to C_0 \to \ldots \to C_n \to \ldots$ such that $C_n \in \mathcal{C}$ for all $n \leq \omega$.

Assume M is $\mathcal{C}$-coresolved. If M has a $\mathcal{C}$–coresolution such that $C_i = 0$ for all $i \geq n+1$, then the least such n (among all such $\mathcal{C}$–coresolutions) is called the $\mathcal{C}$–*coresolution dimension* of M. Otherwise M is said to have $\mathcal{C}$–coresolution dimension ∞.

Clearly any module is $\mathcal{P}_0$–resolved, and the $\mathcal{P}_0$–resolution dimension is exactly the projective dimension. Similarly, any module is $\mathcal{I}_0$–coresolved; the $\mathcal{I}_0$–coresolution dimension is exactly the injective dimension.

Now we continue with several basic properties of tilting cotorsion pairs:

Lemma 3.8. *Let R be a ring and T be an n–tilting module. Denote by $\mathfrak{T} = (\mathcal{A}, \mathcal{B})$ the n–tilting cotorsion pair induced by T.*

(a) *Let $0 \to P_n \to \ldots \to P_0 \to T \to 0$ be a projective resolution of T with the syzygy modules $S_0 = T, \ldots, S_n = P_n$. Let $S = \bigoplus_{i \le n} S_i$. Then $\mathfrak{T}$ is the cotorsion pair generated by S. In particular, $\mathfrak{T}$ is complete.*

(b) *$\mathcal{A} \subseteq \mathcal{P}_n$ and $\mathcal{B} \subseteq \mathrm{Gen}(T)$.*

Each of the short exact sequences forming the long exact sequence in (T3) is given by a special $\mathcal{B}$-preenvelope of an element of $\mathcal{A}$. The length r in (T3) can be taken $\le n$.

(c) *The kernel of $\mathfrak{T}$ equals* $\mathrm{Add}(T)$*.*

(d) *Each $M \in \mathcal{B} \cap \mathcal{P}_n$ has* $\mathrm{Add}(T)$*–resolution dimension $\le n$.*

Proof. (a) This follows by Theorem 1.40 (b).

(b) By assumption, $S \in \mathcal{P}_n$, so $\mathcal{A} \subseteq \mathcal{P}_n$ by Theorem 2.12.
Let $M \in \mathcal{B}$. Consider the long exact sequence from (T3):

$$0 \to R \xrightarrow{\varphi} T_0 \xrightarrow{\varphi_0} T_1 \xrightarrow{\varphi_1} \ldots \xrightarrow{\varphi_{r-1}} T_r \xrightarrow{\varphi_r} 0.$$

Since $T_i \in \mathcal{A}$ for all $i \le r$, and $\mathcal{A}$ is resolving by Lemma 1.20, we have $K_i = \mathrm{Ker}(\varphi_i) \in \mathcal{A}$. In particular, $K_i \in \mathcal{P}_n$. Let $f : R^{(\lambda)} \to M$ be an epimorphism and put $g = \varphi^{(\lambda)}$. Consider the exact sequence $0 \to R^{(\lambda)} \xrightarrow{g} T_0^{(\lambda)} \to K_1^{(\lambda)} \to 0$, and form the pushout of f and g:

$$\begin{tikzcd}
0 \arrow[r] & R^{(\lambda)} \arrow[r, "g"] \arrow[d, "f"] & T_0^{(\lambda)} \arrow[r] \arrow[d, "h"] & K_1^{(\lambda)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & K_1^{(\lambda)} \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

Since $M \in \mathcal{B}$, the second row splits, so M is a direct summand in G. Since h is surjective, $G \in \mathrm{Gen}(T_0) \subseteq \mathrm{Gen}(T)$, and $M \in \mathrm{Gen}(T)$. This proves that $\mathcal{B} \subseteq \mathrm{Gen}(T)$.

By (T2), $T_i \in \mathrm{Add}(T) \subseteq \mathcal{B}$ for all $i \le r$. So the embedding $K_i \hookrightarrow T_i$ is a special $\mathcal{B}$-preenvelope of $K_i \in \mathcal{A}$, and $\operatorname{proj\,dim} K_i \le n$, for each $i \le r$. If $n < r$, then the short exact sequence $0 \to K_n \to T_n \to K_{n+1} \to 0$ splits, since $\mathrm{Ext}^1_R(K_{n+1}, K_n) \cong \cdots \cong \mathrm{Ext}^n_R(K_{n+1}, K_1) \cong \mathrm{Ext}^{n+1}_R(K_{n+1}, K_0) = 0$. So we can assume $r \le n$ in (T3).

(c) By (T2), $\mathrm{Add}(T) \subseteq \mathcal{A} \cap \mathcal{B}$.
Conversely, let $M \in \mathcal{A} \cap \mathcal{B}$. By part (b), $M \in \mathrm{Gen}(T)$. So the canonical map $\varphi \in \mathrm{Hom}_R(T^{(\mathrm{Hom}_R(T,M))}, M)$ is surjective, and there is a short exact sequence

$$0 \to L \to T^{(\mathrm{Hom}_R(T,M))} \xrightarrow{\varphi} M \to 0. \tag{3.2}$$

Applying $\mathrm{Hom}_R(T,-)$ to (3.2), we obtain the long exact sequence

$$0 \to \mathrm{Hom}_R(T,L) \to \mathrm{Hom}_R(T, T^{(\mathrm{Hom}_R(T,M))}) \overset{\mathrm{Hom}_R(T,\varphi)}{\to} \mathrm{Hom}_R(T,M) \to$$
$$\mathrm{Ext}^1_R(T,L) \to \mathrm{Ext}^1_R(T, T^{(\mathrm{Hom}_R(T,M))}) \to \mathrm{Ext}^1_R(T,M) \to \dots$$
$$\mathrm{Ext}^i_R(T,L) \to \mathrm{Ext}^i_R(T, T^{(\mathrm{Hom}_R(T,M))}) \to \mathrm{Ext}^i_R(T,M) \to \dots .$$

By definition, $\mathrm{Hom}_R(T,\varphi)$ is surjective, so $\mathrm{Ext}^1_R(T,L) = 0$ by (T2). We also have $\mathrm{Ext}^i_R(T,M) = 0$ for all $0 < i < \omega$, so condition (T2) implies that $L \in T^{\perp_\infty} = \mathcal{B}$. Since $M \in \mathcal{A}$, (3.2) splits, and $M \in \mathrm{Add}(T)$.

(d) Let $M \in \mathcal{B} \cap \mathcal{P}_n$. An iteration of special $\mathcal{A}$-precovers (of M etc.) gives rise to a long exact sequence

$$0 \to K_n \to E_n \xrightarrow{\psi_n} E_{n-1} \xrightarrow{\psi_{n-1}} \dots \xrightarrow{\psi_1} E_0 \xrightarrow{\psi_0} M \to 0,$$

where $E_i \in \mathrm{Add}(T)$, $K_i = \mathrm{Ker}\,\psi_i \in \mathcal{B}$ and ψ_i induces a special $\mathcal{A}$-precover of its image for all $i \leq n$. By assumption, $M \in \mathcal{P}_n$, so $\mathrm{Ext}^1_R(K_{n-1}, K_n) \cong \dots \cong \mathrm{Ext}^n_R(K_0, K_n) \cong \mathrm{Ext}^{n+1}_R(M, K_n) = 0$. It follows that $K_{n-1} \in \mathrm{Add}(T)$, so we can take $E_n = K_{n-1}$ and $K_n = 0$. ∎

If T is a tilting module, then its projective dimension is the maximum of projective dimensions of the modules in $\mathcal{A} = {}^{\perp}(T^{\perp_\infty})$. In particular, by Lemma 3.8 (b), equivalent tilting modules have equal projective dimensions.

By Lemma 3.8 (c), the kernel of $\mathfrak{T}$ equals $\mathrm{Add}(T)$. So the classes $\mathcal{A}$ and $\mathcal{B}$ can be recovered from the kernel simply using the equalities $\mathcal{B} = (\mathrm{Add}(T))^{\perp}$ and $\mathcal{A} = {}^{\perp}\mathcal{B}$.

There is another way of recovering $\mathcal{A}$ and $\mathcal{B}$ from the kernel, via $\mathrm{Add}(T)$–resolutions and $\mathrm{Add}(T)$–coresolutions in the sense of Definition 3.7:

Proposition 3.9. *Let R be a ring, T be an n–tilting module, and $(\mathcal{A}, \mathcal{B})$ the n–tilting cotorsion pair induced by T.*

- (a) *$\mathcal{A}$ coincides with the class of all* $\mathrm{Add}(T)$*–coresolved modules of* $\mathrm{Add}(T)$*–coresolution dimension $\leq n$.*
- (b) *$\mathcal{B}$ coincides with the class of all* $\mathrm{Add}(T)$*–resolved modules. In particular, $\mathcal{B}$ is closed under direct sums.*

Proof. (a) Since $\mathcal{A}$ is resolving, $M \in \mathcal{A}$ for any module M of finite $\mathrm{Add}(T)$–resolution dimension.

Conversely, let $A \in \mathcal{A}$. An iteration of special $\mathcal{B}$–preenvelopes (of A etc.) yields a long exact sequence

$$0 \to A \to E_0 \xrightarrow{\psi_0} E_1 \xrightarrow{\psi_1} \dots \xrightarrow{\psi_{n-1}} E_n \xrightarrow{\psi_n} K_{n+1} \to 0,$$

where $E_i \in \mathrm{Add}(T)$ for all $i \leq n$ and $K_{n+1} \in \mathcal{A}$. Let $K_i = \mathrm{Ker}\,\psi_i$ $(i \leq n)$. By Lemma 3.8 (b), $K_{n+1} \in \mathcal{P}_n$, so $\mathrm{Ext}^1_R(K_{n+1}, K_n) \cong \dots \cong \mathrm{Ext}^n_R(K_{n+1}, K_1) \cong$

$\mathrm{Ext}_R^{n+1}(K_{n+1}, A) = 0$. It follows that $K_{n+1} \in \mathrm{Add}(T)$, so we can take $E_n = K_n$ and $K_{n+1} = 0$.

(b) If $M \in \mathcal{B}$, then an $\mathrm{Add}(T)$–resolution is obtained by an iteration of special $\mathcal{A}$-precovers (of M etc.).

Conversely, assume there exists an $\mathrm{Add}(T)$–resolution B

$$\ldots \to E_n \to \ldots \to E_0 \to B \to 0.$$

Denote by K_0 the kernel of the epimorphism $E_0 \to B$, by K_1 the kernel of the epimorphism $E_1 \to K_0$, etc. Let $A \in \mathcal{A}$. Then $\mathrm{Ext}_R^1(A, B) \cong \mathrm{Ext}_R^2(A, K_0) \cong \cdots \cong \mathrm{Ext}_R^{n+1}(A, K_{n-1}) = 0$ by Lemma 3.8 (b), so $B \in \mathcal{B}$. ∎

Indeed, we can do slightly better:

Corollary 3.10. *Let R be a ring, T be an n–tilting module, and $(\mathcal{A}, \mathcal{B})$ the n–tilting cotorsion pair induced by T.*

(a) *$\mathcal{A}$ coincides with the class of all modules A possessing an exact sequence*

$$0 \to A \to T^{(\kappa_0)} \to \ldots \to T^{(\kappa_n)} \to 0$$

where κ_i is a cardinal for each $i \leq n$.

(b) *$\mathcal{B}$ coincides with the class, $\mathrm{Gen}_n(T)$, consisting of all modules B possessing an exact sequence*

$$T^{(\lambda_n)} \to \ldots \to T^{(\lambda_1)} \to B \to 0,$$

where λ_i is a cardinal for each $1 \leq i \leq n$.

Proof. (a) This follows by possibly adding elements of $\mathrm{Add}(T)$ to the middle terms of the short exact sequence forming the $\mathrm{Add}(T)$–coresolution characterizing $A \in \mathcal{A}$ in Proposition 3.9 (a).

(b) As in part (a), we infer from Proposition 3.9 (b) that $B \in \mathcal{B}$, iff B possesses a long exact sequence

$$\ldots \to T^{(\lambda_i)} \to \ldots \to T^{(\lambda_1)} \to B \to 0,$$

where λ_i is a cardinal for each $1 \leq i < \omega$. So clearly, $\mathcal{B} \subseteq \mathrm{Gen}_n(T)$.

Conversely, if $B \in \mathrm{Gen}_n(T)$ possesses a sequence

$$T^{(\lambda_n)} \xrightarrow{f_n} \ldots \xrightarrow{f_2} T^{(\lambda_1)} \xrightarrow{f_1} B \to 0,$$

then $\mathrm{Ext}_R^i(T, B) \cong \mathrm{Ext}_R^{i+1}(T, \mathrm{Ker}(f_1)) \cong \cdots \cong \mathrm{Ext}_R^{i+n}(T, \mathrm{Ker}(f_n)) = 0$ for each $i \geq 1$ since $\operatorname{proj\,dim} T \leq n$. So $B \in T^{\perp_\infty} = \mathcal{B}$. ∎

By Corollary 3.10 (b), any tilting module of projective dimension n satisfies $T^{\perp_\infty} = \mathrm{Gen}_n(T)$. Also the converse holds, so T is an n–tilting module, if and only if $T^{\perp_\infty} = \mathrm{Gen}_n(T)$ (see [18]).

The classes $X^{\perp_\infty}$ and $\mathrm{Gen}_n(X)$ are well–defined for any object X of a cocomplete abelian category $\mathcal{X}$. So the condition $X^{\perp_\infty} = \mathrm{Gen}_n(X)$ is a suitable defining condition for an (infinite–dimensional) tilting object X of $\mathcal{X}$ that avoids the problem of the possible non–existence of projective objects in $\mathcal{X}$.

The equivalence of tilting modules defined above can be expressed in a simpler way:

Lemma 3.11. *Let R be a ring and T_1, T_2 be tilting modules. Then T_1 is equivalent to T_2, iff $\mathrm{Add}(T_1) = \mathrm{Add}(T_2)$, iff $\mathrm{Add}(T_1) \subseteq \mathrm{Add}(T_2)$.*

Proof. If $T_1^{\perp_\infty} = T_2^{\perp_\infty}$, then also ${}^\perp(T_1^{\perp_\infty}) = {}^\perp(T_2^{\perp_\infty})$, and hence $\mathrm{Add}(T_1) = \mathrm{Add}(T_2)$ by Lemma 3.8 (c).

Assume $T_1 \in \mathrm{Add}(T_2)$. Then $\mathcal{B}_{T_2} = T_2^{\perp_\infty} \subseteq T_1^{\perp_\infty} = \mathcal{B}_{T_1}$. However, $\mathcal{B}_{T_1} \subseteq \mathcal{B}_{T_2}$ by Proposition 3.9 (b), so T_1 and T_2 are equivalent. ∎

The following result, due to Angeleri Hügel and Coelho [3], gives a characterization of tilting classes of modules:

Theorem 3.12. *Let R be a ring, $n < \omega$, and $\mathcal{C}$ be a class of modules. Then the following assertions are equivalent:*

(a) *$\mathcal{C}$ is n–tilting.*

(b) *$\mathcal{C}$ is coresolving, special preenveloping, closed under direct sums and direct summands, and ${}^\perp\mathcal{C} \subseteq \mathcal{P}_n$.*

Proof. (a) implies (b): this follows from parts (a) and (b) of Lemma 3.8, and from Proposition 3.9 (b).

(b) implies (a): first the special $\mathcal{C}$–preenvelope of any injective module splits. Since $\mathcal{C}$ is closed under direct summands and it is coresolving, we have $\mathcal{I}_0 \subseteq \mathcal{C}$ and $\mathcal{C}$ is cosyzygy closed. So ${}^{\perp_\infty}\mathcal{C} = {}^\perp\mathcal{C}$ by Lemma 1.21.

The special $\mathcal{C}$-preenvelope of R gives rise to a short exact sequence $0 \to K_0 \to T_0 \to K_1 \to 0$, where $K_0 = R$, $T_0 \in \mathcal{C}$ and $K_1 \in {}^\perp\mathcal{C} \subseteq \mathcal{P}_n$. Since $R \in {}^\perp\mathcal{C}$, we have $T_0 \in \mathcal{C} \cap {}^\perp\mathcal{C}$. By induction we obtain short exact sequences $0 \to K_i \to T_i \to K_{i+1} \to 0$ with $T_i \in \mathcal{C} \cap {}^\perp\mathcal{C}$ and $K_{i+1} \in {}^\perp\mathcal{C} \subseteq \mathcal{P}_n$. Since $K_{n+1} \in \mathcal{P}_n$, the sequence $0 \to K_n \to T_n \to K_{n+1} \to 0$ splits by dimension shifting. So we can assume that $K_{n+1} = 0$, and form the long exact sequence (with $T_i \in \mathcal{C} \cap {}^\perp\mathcal{C}$ for all $i \leq n$)

$$0 \to R \xrightarrow{\varphi_0} T_0 \xrightarrow{\varphi_1} T_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{n-1}} T_{n-1} \xrightarrow{\varphi_n} T_n \to 0. \tag{3.3}$$

Put $T = \bigoplus_{i \leq n} T_i$. We will prove that T is n–tilting. First $T \in \mathcal{C} \cap {}^\perp\mathcal{C} \subseteq \mathcal{P}_n$, so (T1) holds. Since $\mathcal{C}$ is closed under direct sums, $T^{(\kappa)} \in \mathcal{C}$ for each cardinal κ, and (T2) holds. The long exact sequence above gives (T3).

Finally, we will prove that $T^{\perp_\infty} = \mathcal{C}$. Since $T \in {}^\perp\mathcal{C}$, clearly $T^{\perp_\infty} \supseteq \mathcal{C}$. Conversely, let $C \in T^{\perp_\infty}$. Consider a special $\mathcal{C}$–preenvelope ψ_0 of C, a special $\mathcal{C}$–preenvelope ψ_1 of $\operatorname{Coker} \varphi_0$ etc. Since $\operatorname{Coker} \psi_{n+1} \in \mathcal{P}_n$, dimension shifting shows that ψ_{n+1} splits. So there is a long exact sequence

$$0 \to C \xrightarrow{\psi_0} D_0 \xrightarrow{\psi_1} D_1 \xrightarrow{\psi_2} \dots \xrightarrow{\psi_{n-1}} D_{n-1} \xrightarrow{\psi_n} D_n \to 0$$

with $D_i \in \mathcal{C} \subseteq T^{\perp_\infty}$ for all $i < n$, and $D_n \in \mathcal{C} \cap {}^\perp\mathcal{C}$. Since $C \in T^{\perp_\infty}$ and $T^{\perp_\infty}$ is coresolving, we get $\operatorname{Coker} \psi_i \in T^{\perp_\infty}$ for all $i \leq n$. It remains to prove that $\mathcal{C} \cap {}^\perp\mathcal{C} \subseteq {}^\perp(T^{\perp_\infty})$ — then ψ_n splits and, by induction, ψ_0 splits, so $C \in \mathcal{C}$.

Let $M \in \mathcal{C} \cap {}^\perp\mathcal{C}$ $(\subseteq T^{\perp_\infty} \cap \mathcal{P}_n)$. By Lemma 3.8 (d), there is a long exact sequence

$$0 \to E_n \to \ldots \to E_0 \xrightarrow{\eta_0} M \to 0,$$

where $E_i \in \mathrm{Add}(T)$ for all $i \leq n$. By the closure properties of $\mathcal{C}$, $\mathrm{Add}(T) \subseteq \mathcal{C} \cap {}^\perp\mathcal{C}$, and $\mathrm{Ker}\,\eta_0 \in \mathcal{C}$. So η_0 splits, and $M \in \mathrm{Add}(T) \subseteq {}^\perp(T^{\perp_\infty})$. ∎

Note that the proof of (b) implies (a) above is constructive: the tilting module T is obtained as $T = \bigoplus_{i \leq n} T_i$ where T_i form the long exact sequence (3.3) obtained by an iteration of special $\mathcal{C}$–preenvelopes, starting from a special $\mathcal{C}$–preenvelope of R, $\varphi_0 : R \to T_0$, over a special $\mathcal{C}$–preenvelope of the cokernel of φ_0, etc.

Now we can characterize tilting cotorsion pairs by the closure properties of their components:

Corollary 3.13. *Let $n < \omega$. Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then the following assertions are equivalent:*

(a) *$\mathfrak{C}$ is an n–tilting cotorsion pair.*

(b) *$\mathfrak{C}$ is a hereditary (and complete) cotorsion pair such that $\mathcal{A} \subseteq \mathcal{P}_n$ and $\mathcal{B}$ is closed under direct sums.*

Proof. (a) implies (b). By Theorem 3.12 for $\mathcal{C} = \mathcal{B}$.

(b) implies (a). In view of Theorem 3.12, it only remains to prove that completeness of $\mathfrak{C}$ follows from the other assumptions on $\mathfrak{C}$. However, this has already been proved in Corollary 2.34. ∎

Our Definition 3.1 of a tilting module admits infinitely generated modules. Indeed, many of the examples of tilting modules presented in this book are far from being finitely generated. There is, however, an implicit finiteness condition hidden in the notion of a tilting module: every tilting module T is of finite type. This says that though T is large, when computing the corresponding tilting class $T^{\perp_\infty}$, we can replace T by a set $\mathcal{S} \subseteq \text{mod–}R$ such that $T^{\perp_\infty} = \mathcal{S}^{\perp_\infty}$.

In particular, $T^{\perp_\infty}$ is a *definable class* of modules, that is, it is closed under direct limits, products, and pure submodules. Another consequence is that tilting modules are classified up to equivalence by the resolving subcategories of finitely presented modules of bounded projective dimension.

Now, we aim at proving these results in several steps following the recent papers [4], [21], [23], [26] and [71]. We will concentrate on the key ideas of the proofs rather than technicalities, so not all steps will be proved in full generality. We refer to [46] for a complete presentation.

We start with the notions of a module, and a class of finite type:

Definition 3.14. Let R be a ring.
Let $\mathcal{C}$ be a class of modules. Then $\mathcal{C}$ is of *finite type* (*countable type*) provided there exist $n < \omega$ and $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ ($\mathcal{S} \subseteq \mathcal{P}_n^{\leq\omega}$) such that $\mathcal{C} = \mathcal{S}^{\perp_\infty}$.
Let T be a module. The T is of *finite type* (of *countable type*) provided the class $T^{\perp_\infty}$ is of finite type (countable type).

Let $\mathcal{C}$ be a class of finite (countable) type and $\mathcal{A} = {}^{\perp}\mathcal{C}\,(= {}^{\perp_\infty}\mathcal{C})$. Then $(\mathcal{A}, \mathcal{C})$ is a hereditary cotorsion pair generated by the class $\mathcal{A}^{<\omega}$ ($\mathcal{A}^{\leq\omega}$), so $\mathcal{S} = \mathcal{A}^{<\omega}$ ($\mathcal{S} = \mathcal{A}^{\leq\omega}$) is the largest possible choice for $\mathcal{S}$ in Definition 3.14.

Any class of finite type is a tilting class, so there is a rich supply of tilting classes and modules available:

Theorem 3.15. *Let R be a ring and $\mathcal{C}$ be a class of finite type. Then $\mathcal{C}$ is tilting and definable.*

Proof. By assumption, there are $n < \omega$ and $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ such that $\mathcal{C} = \mathcal{S}^{\perp_\infty}$. Clearly $\mathcal{C}$ is closed under direct products. By Lemma 1.34, $\mathcal{C}$ is also closed under direct limits. Since $F^{\perp}$ is closed under pure submodules for any finitely presented module F, $\mathcal{C}$ is closed under pure submodules. So $\mathcal{C}$ is a definable class.

Let $\mathfrak{C} = (\mathcal{A}, \mathcal{C})$ be the cotorsion pair cogenerated by $\mathcal{C}$. By Theorem 1.40, $\mathfrak{C}$ is complete. By Theorem 2.12, $\mathcal{A} \subseteq \mathcal{P}_n$. By Corollary 3.13, $\mathfrak{C}$ is an n–tilting cotorsion pair, that is, $\mathcal{C}$ is an n–tilting class. ∎

The converse of Theorem 3.15 also holds: all tilting classes and modules are of finite type. This is proved in three steps.

The first step shows that all tilting classes are of countable type:

Theorem 3.16. *Let R be a ring, T be a tilting module, and $(\mathcal{A}, \mathcal{B})$ the cotorsion pair induced by T. Then T and $\mathcal{B}$ are of countable type. Moreover, each module $A \in \mathcal{A}$ is $\mathcal{A}^{\leq\omega}$–filtered.*

Proof. First, assume that R is an $\aleph_0$–noetherian ring. We have $\mathcal{A} \subseteq \mathcal{P}_n$ by Lemma 3.8 (b). So the assertion in this case is a particular instance of Theorem 2.33 for $\mu = \aleph_0$.

For the general case, we refer to [71] or [46]. ∎

In order to proceed from the countable type to the finite one, we will first prove the definability of tilting classes. For that purpose, we will need a couple of results concerning countably presented modules and their Ext–orthogonal classes.

We start by recalling the canonical presentation of countably presented modules going back to Bass:

Let $A \in \text{Mod–}R$ be countably presented. Then there exist finitely presented modules $(A_i \mid i < \omega)$, and R–homomorphisms $h_i : A_i \to A_{i+1}$ such that A is the direct limit of the well–ordered direct system

$$A_0 \xrightarrow{h_0} A_1 \xrightarrow{h_1} \dots \xrightarrow{h_{i-1}} A_i \xrightarrow{h_i} A_{i+1} \xrightarrow{h_{i+1}} \dots$$

and the following sequence is pure–exact

$$0 \to \bigoplus_{i<\omega} A_i \xrightarrow{\psi} \bigoplus_{i<\omega} A_i \to A \to 0, \tag{3.4}$$

where ψ is defined by $\psi\epsilon_i = \epsilon_i - \epsilon_{i+1}h_i$, and $\epsilon_i : A_i \to \bigoplus_{i<\omega} A_i$ is the i–th canonical monomorphism for each $i < \omega$.

Let B be a module. For each $i < \omega$, denote by $\xi_i : B^{(\omega)} \to B$ the i–th canonical projection. For each homomorphism $\beta \in \mathrm{Hom}_R(\bigoplus_{i<\omega} A_i, B^{(\omega)})$, we define $\beta_{ji} = \xi_j \beta \epsilon_i$ for all $i, j < \omega$.

An R–homomorphism $\gamma : \bigoplus_{i<\omega} A_i \to B^{(\omega)}$ is called *diagonal*, provided that $\gamma_{ij} = 0$ for all $i \neq j < \omega$.

We will say that ψ has *B–factorization property*, provided that each diagonal map $\gamma \in \mathrm{Hom}_R(\bigoplus_{i<\omega} A_i, B^{(\omega)})$ has a factorization through ψ, that is, there exists $\phi \in \mathrm{Hom}_R(\bigoplus_{i<\omega} A_i, B^{(\omega)})$ such that $\gamma = \phi\psi$.

We will prove that the B–factorization property is closely related to the Mittag–Leffler condition defined below. For that purpose, we introduce further convenient notation: an exact sequence of modules of the form

$$\mathcal{G} : \quad \dots \overset{g_{i+1}}{\to} G_{i+1} \overset{g_i}{\to} G_i \overset{g_{i-1}}{\to} \dots \overset{g_1}{\to} G_1 \overset{g_0}{\to} G_0$$

is called a *tower* of modules. Putting $g_{ij} = g_i \dots g_j$ for all $i < j$ and $g_{ii} = \mathrm{id}_{G_i}$, we obtain an inverse system of modules induced by the tower $\mathcal{G}$, and denoted by $\mathcal{G}' = (G_i, g_{ij} \mid i < j < \omega)$.

We will say that the inverse system $\mathcal{G}'$ (or the tower $\mathcal{G}$) satisfies the *Mittag–Leffler condition*, provided that for each $k < \omega$ there exists $j \geq k$ such that $\mathrm{Im}(g_{ki}) = \mathrm{Im}(g_{kj})$ for all $j \leq i < \omega$, that is, the descending chain of the images of the groups G_{i+1} in G_k is stationary.

Define a map $\nabla_G : \prod_{i<\omega} G_i \to \prod_{i<\omega} G_i$ by the assignment

$$(\dots, a_i, \dots, a_0) \mapsto (\dots, a_i - g_i(a_{i+1}), \dots, a_0 - g_0(a_1)).$$

Then clearly $\mathrm{Ker}(\nabla_G) = \varprojlim_{i<\omega} G_i$ is the inverse limit of the inverse system $\mathcal{G}'$.

If $0 \to \mathcal{G}' \to \mathcal{H}' \to \mathcal{K}' \to 0$ is a short exact sequence of inverse systems induced by towers of modules, then the Snake Lemma from classical homological algebra (see e.g. [40, §1.2]) yields the exact sequence

$$\begin{aligned} 0 \to \varprojlim_{i<\omega} G_i \to \varprojlim_{i<\omega} H_i \to \varprojlim_{i<\omega} K_i \\ \to \mathrm{Coker}(\nabla_G) \to \mathrm{Coker}(\nabla_H) \to \mathrm{Coker}(\nabla_K) \to 0. \end{aligned} \tag{3.5}$$

In particular, if $\mathrm{Coker}(\nabla_G) = 0$, then $\varprojlim$ preserves the exactness of all short exact sequence of inverse systems induced by towers of modules which start from $\mathcal{G}'$. The Mittag–Leffler condition is a sufficient condition for this to happen:

Lemma 3.17. *Assume that $\mathcal{G}$ satisfies the Mittag–Leffler condition. Then* $\mathrm{Coker}(\nabla_G) = 0$.

Proof. First assume that for each $k < \omega$ there is $j > k$ such that $g_{kj} = 0$. Consider a sequence $\bar{x} = (x_k \mid k < \omega) \in \prod_{k<\omega} G_k$. Let $\bar{y} = (y_k \mid k < \omega)$, where $y_k = x_k + c_{k+1} + \dots + c_{j-1}$ and $c_i = g_{ki}(x_i)$ for $j > i > k$. Then $\nabla_G(\bar{y}) = \bar{x}$, so ∇_G is surjective, and $\mathrm{Coker}(\nabla_G) = 0$.

In the general case, let $I_k \subseteq G_k$ be the (stabilized) image of $g_{ki} : G_{i+1} \to G_k$ in G_k (for $k < j \leq i$). Consider the tower

$$\mathcal{I} : \quad \dots \overset{g_{i+1}}{\to} I_{i+1} \overset{g_i}{\to} \dots \overset{g_1}{\to} I_1 \overset{g_0}{\to} I_0.$$

Since all the maps $g_i : I_{i+1} \to I_i$ are surjective, ∇_I is easily seen to be surjective, so $\mathrm{Coker}(\nabla_I) = 0$.

Now the tower

$$\mathcal{J}: \quad \dots \stackrel{\bar{g_{i+1}}}{\to} G_{i+1}/I_{i+1} \stackrel{\bar{g_i}}{\to} \dots \stackrel{\bar{g_1}}{\to} G_1/I_1 \stackrel{\bar{g_0}}{\to} G_0/I_0$$

has the property that for each $i < \omega$ there is $j > i$ such that $\bar{g_{ij}} = 0$. By the first part of the proof, ∇_J is surjective, and $\mathrm{Coker}(\nabla_J) = 0$.

Finally, (3.5) yields the exact sequence

$$0 = \mathrm{Coker}(\nabla_I) \to \mathrm{Coker}(\nabla_G) \to \mathrm{Coker}(\nabla_J) = 0$$

proving that $\mathrm{Coker}(\nabla_G) = 0$. ∎

In fact, if $\mathcal{G}$ is a tower of modules, then besides the obvious equality $\varprojlim \mathcal{G} = \mathrm{Ker}(\nabla_G)$, also $\varprojlim^1 \mathcal{G} = \mathrm{Coker}(\nabla_G)$ and $\varprojlim^i \mathcal{G} = 0$ for $i > 1$, and (3.5) is just the long exact sequence for the derived functors of the left–exact functor $\varprojlim$ (for more details, we refer to [78, §3.5]).

Notice that, if A is a countably presented module with the presentation (3.4), and B is any module, then the inverse system induced by the tower of abelian groups

$$\dots \stackrel{\mathrm{Hom}_R(h_{i+1},B)}{\to} \mathrm{Hom}_R(A_{i+1}, B) \stackrel{\mathrm{Hom}_R(h_i,B)}{\to} \dots \stackrel{\mathrm{Hom}_R(h_0,B)}{\to} \mathrm{Hom}_R(A_0, B)$$

satisfies the Mittag–Leffler condition, if and only if, for each $i < \omega$, the chain of subgroups of $\mathrm{Hom}_R(A_i, B)$

$$\mathrm{Hom}_R(A_{i+1}, B)h_i \supseteq \dots \supseteq \mathrm{Hom}_R(A_{i+j}, B)h_{i+j-1} \dots h_i \supseteq \cdots$$

is stationary.

First we will need the following necessary condition for a diagonal map to factorize through ψ:

Lemma 3.18. *Let R be a ring, A a countably presented module with the presentation (3.4), and B be a module. Assume that $\gamma \in \mathrm{Hom}_R(\bigoplus_{i<\omega} A_i, B^{(\omega)})$ is a diagonal map which has a factorization, ϕ, through ψ. Then there exists a sequence of natural numbers $(l(m) \mid m < \omega)$ such that, for each $m < \omega$, $l(m) > m$, and $\gamma_{kk} h_{k-1} h_{k-2} \dots h_m = -\phi_{k,k+1} h_k h_{k-1} \dots h_m$ for all $k \geq l(m)$.*

Proof. Fix $m \geq 0$. For each $j < \omega$, we have $\phi_{ij} = 0$ and $\psi_{ij} = 0$ for almost all $i < \omega$, since A_j is finitely generated. Let $l(m) > m$ be the least index such that $\phi_{km} = 0$ for all $k \geq l(m)$.

Since γ is diagonal and $\gamma = \phi\psi$, $\sum_{k<\omega} \phi_{ik}\psi_{kj} = 0$ for $i \neq j < \omega$, and $\sum_{k<\omega} \phi_{ik}\psi_{ki} = \gamma_{ii}$ for each $i < \omega$. The former equation yields, for $k > m$, that $\phi_{kj} = \phi_{k,j+1} h_j$ for each $m \leq j < k$, hence $\phi_{km} = \phi_{kk} h_{k-1} h_{k-2} \dots h_m$. The latter equation gives $\phi_{kk} = \phi_{k,k+1} h_k + \gamma_{kk}$. Altogether, we have

$$-\phi_{k,k+1} h_k h_{k-1} \dots h_m = \gamma_{kk} h_{k-1} h_{k-2} \dots h_m$$

for each $k \geq l(m)$. ∎

The following lemma relates the B–factorization property to the Mittag–Leffler condition:

Lemma 3.19. *Let R be a ring, and A be a countably presented module with the presentation (3.4). Let B be a module such that ψ has B–factorization property. Then for each $i < \omega$, the chain of subgroups of* $\operatorname{Hom}_R(A_i, B)$

$$\operatorname{Hom}_R(A_{i+1}, B)h_i \supseteq \cdots \supseteq \operatorname{Hom}_R(A_{i+j}, B)h_{i+j-1} \dots h_i \supseteq \cdots \tag{3.6}$$

is stationary.

Proof. Assume there is $i < \omega$ such that the chain (3.6) is not stationary. So there is an infinite set $S \subseteq \omega$ such that for each $j \in S$ there is $f_j \in \operatorname{Hom}_R(A_{i+j}, B)$ with $f_j h_{i+j-1} \dots h_i \notin \operatorname{Hom}_R(A_{i+j+1}, B)h_{i+j} \dots h_i$. Define a diagonal morphism $\gamma : \bigoplus_{j<\omega} A_j \to B^{(\omega)}$ by $\gamma_{i+j,i+j} = f_j$. By assumption, γ has a factorization, ϕ, through ψ. By Lemma 3.18, this implies that for all $k \geq l(i)$,

$$\gamma_{kk} h_{k-1} h_{k-2} \dots h_i = -\phi_{k,k+1} h_k h_{k-1} \dots h_i.$$

For $j \in S$ with $k = i + j \geq l(i)$, we have $\phi_{k,k+1} \in \operatorname{Hom}_R(A_{i+j+1}, B)$, in contradiction with the choice of f_j. ∎

Also the converse of Lemma 3.19 holds: if the chain (3.6) is stationary for all $i < \omega$, then ψ has B–factorization property (for a proof, we refer to [23, §3]).

An important property of the Mittag–Leffler condition is that it behaves well with respect to pure submodules:

Lemma 3.20. *Let R be a ring, and A be a countably presented module with the presentation (3.4). Let B' be a pure submodule of a module B. Assume that the system of abelian groups induced by the tower* $(\operatorname{Hom}_R(A_i, B), \operatorname{Hom}_R(h_i, B) \mid i < \omega)$ *satisfies the Mittag–Leffler condition. Then so does the inverse system induced by the tower* $(\operatorname{Hom}_R(A_i, B'), \operatorname{Hom}_R(h_i, B') \mid i < \omega)$.

Proof. We have to prove that the chain of subgroups of $\operatorname{Hom}_R(A_i, B')$

$$\operatorname{Hom}_R(A_{i+1}, B')h_i \supseteq \cdots \supseteq \operatorname{Hom}_R(A_{i+j}, B')h_{i+j-1} \dots h_i \supseteq \cdots$$

is stationary. Since the analogous chain with B' replaced by B is stationary by assumption, it suffices to prove that

$$\nu \operatorname{Hom}_R(A_{i+j}, B')f = \operatorname{Hom}_R(A_{i+j}, B)f \cap \nu \operatorname{Hom}_R(A_i, B'),$$

where $\nu : B' \hookrightarrow B$ is the inclusion map and $f = h_{i+j-1} \dots h_i$. The inclusion $\subseteq$ is clear, so it is enough to prove that given any homomorphisms $x \in \operatorname{Hom}_R(A_{i+j}, B)$ and $y \in \operatorname{Hom}_R(A_i, B')$ satisfying $xf = \nu y$, there exists $z \in \operatorname{Hom}_R(A_{i+j}, B')$ such that $y = zf$.

Consider a presentation $0 \to K \xrightarrow{\subseteq} R^p \xrightarrow{\rho} A_{i+j} \to 0$ with K finitely generated and $p < \omega$. Let $(1_m \mid m < p)$ be the canonical basis of R^p, and $k_n = \sum_{m<p} 1_m r_{nm}$ $(n < q)$ be an R–generating subset of K. Let $d_l = \sum_{m<p} 1_m s_{lm}$ $(l < t)$ be a finite set of elements of R^p such that $\rho(d_l) = f(a_l)$, where $\{a_l \mid l < t\}$ is a finite R–generating subset of A_i.

The existence of the map x implies solvability in B of the following system of R–linear equations in variables x_m $(m < p)$:

$$\sum_{m<p} x_m r_{nm} = 0 \ (n < q), \tag{3.7}$$

$$\sum_{m<p} x_m s_{lm} = y(a_l) \ (l < t). \tag{3.8}$$

Since B' is pure in B, there is a solution, $(b_0, \dots, b_{p-1})$ of this system in B'. Define a map $w : R^p \to B'$ by $w(1_m) = b_m$ $(m < p)$. Then $w \restriction K = 0$ by (3.7), so w induces $z \in \mathrm{Hom}_R(A_{i+j}, B')$ such that $w = z\rho$. Finally, $y = zf$ by (3.8). ■

Now we can prove

Theorem 3.21. *Let R be a ring, A a countably presented module, and $\mathcal{B} = A^{\perp}$. Assume that $B^{(\omega)} \in \mathcal{B}$ for each $B \in \mathcal{B}$. Then $\mathcal{B}$ is closed under pure submodules.*

Proof. Consider the presentation of A from (3.4). Let $B \in \mathcal{B}$ and B' be a pure submodule of B. By assumption, $\mathrm{Ext}^1_R(A, B^{(\omega)}) = 0$, so ψ clearly has B–factorization property. By Lemmas 3.19 and 3.20, this implies that the inverse system induced by the tower $(\mathrm{Hom}_R(A_i, B'), \mathrm{Hom}_R(h_i, B') \mid i < \omega)$ is Mittag–Leffler.

Consider the pure–exact sequence

$$\mathcal{X} \ : \ 0 \to B' \to B \xrightarrow{\rho} B/B' \to 0.$$

Since all modules A_i $(i < \omega)$ are finitely presented, an application of $\mathrm{Hom}_R(A_i, -)$ $(i < \omega)$ to $\mathcal{X}$ yields an inverse system of short exact sequences

$$0 \to \mathrm{Hom}_R(A_i, B') \to \mathrm{Hom}_R(A_i, B) \to \mathrm{Hom}_R(A_i, B/B') \to 0 \quad (i < \omega).$$

However, $(\mathrm{Hom}_R(A_i, B'), \mathrm{Hom}_R(h_i, B') \mid i < \omega)$ satisfies the Mittag–Leffler condition, so Lemma 3.17 gives exactness of the sequence

$$0 \to \varprojlim_{i<\omega} \mathrm{Hom}_R(A_i, B') \to \varprojlim_{i<\omega} \mathrm{Hom}_R(A_i, B) \to \varprojlim_{i<\omega} \mathrm{Hom}_R(A_i, B/B') \to 0$$

and hence of

$$0 \to \mathrm{Hom}_R(A, B') \to \mathrm{Hom}_R(A, B) \xrightarrow{\mathrm{Hom}_R(A,\rho)} \mathrm{Hom}_R(A, B/B') \to 0.$$

In particular, $\mathrm{Hom}_R(A, \rho)$ is surjective, and $\mathrm{Ext}^1_R(A, B) = 0$ by assumption, so we conclude that $\mathrm{Ext}^1_R(A, B') = 0$. ■

As an immediate corollary, we obtain:

Corollary 3.22. *Let R be a ring, T be a tilting module, and $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair induced by T. Then $\mathcal{B}$ is definable.*

Proof. Clearly $\mathcal{B}$ is coresolving, and closed under direct products. By Proposition 3.9 (b), $\mathcal{B}$ is closed under direct sums. Since the canonical map of a direct sum onto a direct limit is a pure–epimorphism, it suffices to prove that $\mathcal{B}$ is closed under pure submodules.

However, $\mathcal{B}$ is of countable type by Theorem 3.16, so the closure of $\mathcal{B}$ under pure submodules follows by Theorem 3.21. ■

In order to refine Theorem 3.16 further to finite type, we will use the following criterion:

Lemma 3.23. *Let R be a ring and T be a tilting module. Let $(\mathcal{A}, \mathcal{B})$ be the tilting cotorsion pair induced by T. Then T is of finite type, iff $\mathcal{A}^{\leq\omega} \subseteq \varinjlim \mathcal{A}^{<\omega}$.*

Proof. If T is of finite type, then $\mathcal{B} = (\mathcal{A}^{<\omega})^{\perp}$ by 3.14, so $\mathcal{A} \subseteq \varinjlim \mathcal{A}^{<\omega}$ by Theorem 2.61.

Conversely, let $\mathcal{T} = (\mathcal{A}^{<\omega})^{\perp}$. Then $\mathcal{B} \subseteq \mathcal{T}$, and both $\mathcal{B}$ and $\mathcal{T}$ are definable (by Corollary 3.22 and Theorem 3.15, respectively). By Lemma 1.38, a module belongs to a definable class, iff its pure–injective envelope does. So it remains to show that $\mathcal{B}$ and $\mathcal{T}$ contain the same pure–injective modules.

Let $M \in \mathcal{T}$ be pure–injective and let $A \in \mathcal{A}$. By Theorem 3.16, A is $\mathcal{A}^{\leq\omega}$-filtered. By assumption, $\mathcal{A}^{\leq\omega} \subseteq \varinjlim \mathcal{A}^{<\omega}$. Since $\varinjlim \mathcal{A}^{<\omega}$ is closed under extensions and direct limits by Theorem 2.61, by induction on the length of a $\mathcal{A}^{\leq\omega}$-filtration of A, we infer that $A \in \varinjlim \mathcal{A}^{<\omega}$.

So there is a direct system $(A_i, f_{ji} \mid i \leq j \in I)$ of modules in $\mathcal{A}^{<\omega}$ such that $A = \varinjlim_{i \in I} A_i$, and $\operatorname{Ext}^j_R(A_i, M) = 0$ for all $i \in I$ and $j > 0$. Since M is pure–injective, Lemma 1.35 gives $\operatorname{Ext}^j_R(A, M) \cong \varprojlim_{i \in I} \operatorname{Ext}^j_R(A_i, M) = 0$ for all $j > 0$, hence $M \in \mathcal{B}$.

This proves that $\mathcal{T}$ contains the same pure–injective modules as $\mathcal{B}$. ■

Already at this point, Lemmas 2.63 and 3.23 yield that each 1–tilting module is of finite type. In order to extend this fact by induction to n–tilting modules, we need one more lemma:

Lemma 3.24. *Let R be a ring, $n > 0$, and T be a tilting module of projective dimension n. Let $\mathfrak{C} = (\mathcal{A}', \mathcal{B}')$ be the cotorsion pair defined by $\mathcal{B}' = (\Omega(T))^{\perp_\infty}$, where $\Omega(T)$ is the first syzygy module of T. Then $\mathfrak{C}$ is a tilting cotorsion pair induced by an $(n-1)$–tilting module T'.*

Proof. Since $\Omega(T)$ has projective dimension $n - 1$, by Corollary 3.13, it suffices to prove that the class $\mathcal{B}'$ is closed under direct sums.

For this purpose, it suffices to show that $\mathcal{B}'$ coincides with the class of all modules M such that there is an exact sequence $0 \to M \to B \to C \to 0$, where $B \in T^{\perp_\infty}$ and $C \in \operatorname{Add}(T)$.

The existence of the exact sequence gives $\operatorname{Ext}^i_R(\Omega(T), M) \cong \operatorname{Ext}^{i+1}_R(T, M) = 0$ for each $i > 0$, because $0 = \operatorname{Ext}^i_R(T, C) \to \operatorname{Ext}^{i+1}_R(T, M) \to \operatorname{Ext}^{i+1}_R(T, B) = 0$ is exact.

Conversely, if $M \in \mathcal{B}'$, then the special $T^{\perp_\infty}$–preenvelope of M yields an exact sequence $0 \to M \to B \to C \to 0$, where $B \in T^{\perp_\infty}$ ($\subseteq \mathcal{B}'$) and $C \in {}^{\perp}(T^{\perp_\infty}) \cap \mathcal{B}'$. Moreover, $0 = \operatorname{Ext}^1_R(T, B) \to \operatorname{Ext}^1_R(T, C) \to \operatorname{Ext}^2_R(T, M) = \operatorname{Ext}^1_R(\Omega(T), M) = 0$ is exact, so $C \in {}^{\perp}(T^{\perp_\infty}) \cap T^{\perp_\infty} = \operatorname{Add}(T)$ by Lemma 3.8 (c). ■

We arrive at the main result of this chapter:

Theorem 3.25. *Let R be a ring, T a tilting module, and $(\mathcal{A}, \mathcal{B})$ the cotorsion pair induced by T. Then*

(a) *T and $\mathcal{B}$ are of finite type.*

(b) *T is equivalent to a tilting module T_{fin} such that T_{fin} is $\mathcal{A}^{<\omega}$–filtered.*

Proof. (a) The proof is by induction on $n = \operatorname{proj\,dim} T$. There is nothing to prove for $n = 0$, since T is then equivalent to R.

Assume $n > 0$ and consider the cotorsion pair $\mathfrak{C} = (\mathcal{A}', \mathcal{B}')$ defined by $\mathcal{B}' = (\Omega(T))^{\perp_\infty}$, where $\Omega(T)$ is the first syzygy module of T. By Lemma 3.24, $\mathfrak{C}$ is a tilting cotorsion pair induced by a tilting module of projective dimension $n-1$, so $\mathcal{B}'$ is of finite type by the inductive premise.

Let $A \in \mathcal{A}^{\leq\omega}$. By Lemma 3.23, in order to prove that T is of finite type, it suffices to show that $A \in \varinjlim \mathcal{A}^{<\omega}$.

By assumption, there is an exact sequence $0 \to K \to F \to A \to 0$ with F countably generated projective and $K \in \mathcal{P}_{n-1}^{\leq\omega}$. For each $j > 0$ and $B' \in \mathcal{B}'$, we have $\operatorname{Ext}_R^{j+1}(T, B') = \operatorname{Ext}_R^j(\Omega(T), B') = 0$. Since $A \in \mathcal{A}$, A is a direct summand in a module filtered by R and by the syzygies of T (see Corollary 1.41 and Lemma 3.8 (a)). We have $0 = \operatorname{Ext}_R^{j+1}(A, B') \cong \operatorname{Ext}_R^j(K, B')$ for all $j > 0$, so $K \in \mathcal{A}'$, and $K \in (\mathcal{A}')^{\leq\omega}$.

Let $\mathcal{C} = (\mathcal{A}')^{<\omega}$ $(\subseteq \mathcal{A}^{<\omega})$. Since $\mathcal{B}'$ is of finite type, K is a direct summand in a $\mathcal{C}$–filtered module P by Corollary 1.41, so $K \oplus L = P$ for a module L. By Theorem 2.20 (for $\kappa = \omega$) and Lemma 2.29, we can, moreover, assume that P is countably presented. Let $G = P^{(\omega)}$. Then $K \oplus G \cong (K \oplus L)^{(\omega)} \cong G$, and there is an exact sequence

$$0 \to G \xrightarrow{\subseteq} H \to A \to 0,$$

where G and $H \cong F \oplus G$ are countably presented and $\mathcal{C}$-filtered. Again by Theorem 2.20, there exist strictly increasing $\mathcal{C}$–filtrations $(G_i \mid i < \omega)$ of G, and $(H_i \mid i < \omega)$ of H. Possibly taking a subfiltration of the latter filtration, we can, moreover, assume that $G_i \subseteq H_i$ for each $i < \omega$. Then $A \cong \varinjlim_{i<\omega} A_i$ where $A_i = H_i/G_i$.

It remains to prove that $A_i \in \mathcal{A}^{<\omega}$ for each $i < \omega$.
First we show that $A_i \in \mathcal{A}$: since $H_i \in \mathcal{A}$, if suffices to extend an arbitrary homomorphism $f \in \operatorname{Hom}_R(G_i, B)$ with $B \in \mathcal{B}$ to some $g \in \operatorname{Hom}_R(H_i, B)$. However, G/G_i is $\mathcal{C}$-filtered, so $G/G_i \in \mathcal{A}' \subseteq \mathcal{A}$. Hence $\operatorname{Ext}_R^1(G/G_i, B) = 0$, and f can be extended to $h \in \operatorname{Hom}_R(G, B)$. Similarly, since $A \in \mathcal{A}$, h extends to some $k \in \operatorname{Hom}_R(H, B)$. Now it suffices to take $g = k \restriction H_i$. This gives $A_i \in \mathcal{A}$.

Finally, we prove that $A_i \in \mathcal{A}^{<\omega}$. Since $\mathcal{A}$ is resolving, it suffices to show that any finitely generated module $M \in \mathcal{A}$ is finitely presented. But M is $\mathcal{A}^{\leq\omega}$–filtered by Theorem 3.16, so Theorem 2.20 and Lemma 2.29 yield that M is countably presented. Hence there is an exact sequence $0 \to N \xrightarrow{\subseteq} R^{(m)} \to M \to 0$, where $m < \omega$ and $N = \bigcup_{j<\omega} N_j$, where N_j are finitely generated submodules of N.

Let E_j denote the injective envelope of N/N_j. Define $f : N \to \prod_{j<\omega} E_j$ by $f(n) = (n + N_j)_{j<\omega}$. Then the image of f is contained in $\bigoplus_{j<\omega} E_j \in \mathcal{B}$. Since $M \in \mathcal{A}$, there is $g \in \operatorname{Hom}_R(R^{(m)}, \bigoplus_{j<\omega} E_j)$ such that $g \restriction N = f$. However, the image of g is finitely generated, so there exists $j < \omega$ such that $N_j = N$ proving that M is finitely presented.
This proves that $A \in \varinjlim \mathcal{A}^{<\omega}$, and hence that T is of finite type.

(b) By part (a), $\mathcal{B} = (\mathcal{A}^{<\omega})^\perp$. By Corollary 1.41 and Lemma 3.8 (c), there are a $\mathcal{A}^{<\omega}$-filtered module T_{fin} and a module $Q \in \operatorname{Add}(T)$ such that $T_{fin} = Q \oplus T$. Then T_{fin} is a tilting module with $T^{\perp_\infty} = T_{fin}^{\perp_\infty}$, so T_{fin} is equivalent to T. ■

Of course, there is an explicit construction of the tilting module T_{fin} available: the proof of the implication (b) implies (a) in Theorem 3.12 shows that *any* iteration of special $\mathcal{B}$–preenvelopes: $\varphi_0 : R \to T_0$ of the ring R, φ_1 of the cokernel of φ_0 etc., yields a long exact sequence

$$0 \to R \overset{\varphi_0}{\to} T_0 \to T_1 \to \ldots \to T_{n-1} \to T_n \to 0$$

such that $T' = \bigoplus_{i \le n} T_i$ is a tilting module equivalent to T. By part (a), $\mathcal{B} = \mathcal{C}^{\perp_\infty}$, where $\mathcal{C} = \mathcal{A}^{<\omega}$. By Theorem 1.40 (a), each of the special $\mathcal{B}$–preenvelopes φ_i above can be taken so that its cokernel is $\mathcal{C}$–filtered. But then also each T_i $(i \le n)$ is $\mathcal{C}$–filtered, and so is T'.

In contrast with Theorem 3.16, T itself need not in general possess an $\mathcal{A}^{<\omega}$–filtration. For example, if $T = R \oplus P$, where P is a countably generated projective module which is not a direct sum of finitely generated projective modules, then T is not $\mathcal{P}_0^{<\omega}$–filtered.

The fact that tilting classes coincide with the classes of finite type makes it possible to classify all tilting classes by the resolving subcategories of mod–R.

A class of modules $\mathcal{S}$ is called a *resolving subcategory* of mod–R, if $\mathcal{P}_0^{<\omega} \subseteq \mathcal{S} \subseteq$ mod-R, $\mathcal{S}$ is closed under extensions and direct summands, and $A \in \mathcal{S}$, whenever there is an exact sequence $0 \to A \to B \to C \to 0$ with $B, C \in \mathcal{S}$ (cf. with Definition 1.18 (i)).

Before characterizing tilting classes by means of resolving subcategories of bounded projective dimension, we note that the property of being a resolving subcategory can always be tested in a simplified form:

Lemma 3.26. *Let R be a ring and $\mathcal{S} \subseteq$ mod–R.*
Then $\mathcal{S}$ is resolving, if and only if $R \in \mathcal{S}$, $\mathcal{S}$ is closed under extensions and direct summands, and $A \in \mathcal{S}$, whenever there is an exact sequence $0 \to A \to P \to C \to 0$ with P finitely generated projective and $C \in \mathcal{S}$.
In particular, if $\mathcal{S} \subseteq \mathcal{P}_1$, then $\mathcal{S}$ is resolving, if and only if $R \in \mathcal{S}$ and $\mathcal{S}$ is closed under extensions and direct summands.

Proof. The only–if part is clear. Conversely, if $R \in \mathcal{S}$ and $\mathcal{S}$ is closed under extensions and direct summands, then clearly $\mathcal{P}_0^{<\omega} \subseteq \mathcal{S}$.

Consider an exact sequence $0 \to A \to B \to C \to 0$ with $B, C \in \mathcal{S}$. By assumption, there is an exact sequence $0 \to K \to P \to C \to 0$ with P finitely generated projective and $K \in \mathcal{S}$. Consider the pullback

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Since $\mathcal{S}$ is closed under extensions, the left–hand column gives $L \in \mathcal{S}$. Since P is projective, the upper row splits, A is a direct summand in L, and hence $A \in \mathcal{S}$. ■

Theorem 3.27. *Let R be a ring and $n < \omega$. There is a bijective correspondence between n–tilting classes of right R–modules, and resolving subcategories $\mathcal{S}$ of mod-R such that $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$. The correspondence is given by the mutually inverse assignments $\mathcal{C} \mapsto ({}^{\perp}\mathcal{C})^{<\omega}$ and $\mathcal{S} \mapsto \mathcal{S}^{\perp}$.*

Proof. Let $\mathcal{C}$ be an n–tilting class. By Theorem 3.25, $\mathcal{C}$ is of finite type, so there exists $\mathcal{T} \subseteq \mathcal{P}_n^{<\omega}$ such that $\mathcal{C} = \mathcal{T}^{\perp_\infty}$. Then clearly $({}^{\perp}\mathcal{C})^{<\omega}$ is a resolving subcategory of mod–R.

Conversely, let $\mathcal{S}$ be a resolving subcategory of mod–R such that $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$. Then $\mathcal{C} = \mathcal{S}^{\perp}$ is a class of finite type, so $\mathcal{C}$ is n–tilting by Theorem 3.15.

Let $\mathcal{C}$ be an n–tilting class, so $\mathcal{C} = \mathcal{T}^{\perp_\infty}$ for class $\mathcal{T} \subseteq \mathcal{P}_n^{<\omega}$. Let $\mathcal{S} = ({}^{\perp}\mathcal{C})^{<\omega}$. Then $\mathcal{C} = ({}^{\perp}\mathcal{C})^{\perp} \subseteq \mathcal{S}^{\perp}$. Conversely, $\mathcal{T} \subseteq \mathcal{S}$, so $\mathcal{S}^{\perp} \subseteq \mathcal{T}^{\perp_\infty} = \mathcal{C}$.

Let $\mathcal{S}$ be a resolving subcategory of mod–R such that $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$. Clearly $\mathcal{S} \subseteq ({}^{\perp}(\mathcal{S}^{\perp}))^{<\omega}$. By Theorem 2.61, ${}^{\perp}(\mathcal{S}^{\perp}) \subseteq {}^{\intercal}(\mathcal{S}^{\intercal}) = \varinjlim \mathcal{S}$. By Lemma 2.55, $\mathcal{S} = (\varinjlim \mathcal{S})^{<\omega}$, so we conclude that $({}^{\perp}(\mathcal{S}^{\perp}))^{<\omega} = \mathcal{S}$. ■

Now, we pause to consider two particular cases where tilting classes and/or modules can be described in more detail. In both cases, we will deal with 1–tilting modules and classes, so we will need the following general result:

Lemma 3.28. *Let R be a ring. A module T is 1–tilting, iff* $\mathrm{Gen}(T) = T^{\perp}$. *In this case* $\mathrm{Pres}(T) = \mathrm{Gen}(T)$.

Proof. Assume that T is 1–tilting. Then $T^{\perp}$ is closed under homomorphic images. Since (T2) says that $T^{(\kappa)} \in T^{\perp}$, we get $\mathrm{Gen}(T) \subseteq T^{\perp}$. By Lemma 3.8 (b), $T^{\perp} \subseteq \mathrm{Gen}(T)$, so $\mathrm{Gen}(T) = T^{\perp}$.

Conversely, assume that $\mathrm{Gen}(T) = T^{\perp}$. Let N be a module and E be its injective hull. Applying $\mathrm{Hom}_R(T, -)$ to $0 \to N \to E \to E/N \to 0$, we get $0 = \mathrm{Ext}^1_R(T, E/N) \to \mathrm{Ext}^2_R(T, N) \to \mathrm{Ext}^2_R(T, E) = 0$, since $T^{\perp}$ is closed under homomorphic images. So $\mathrm{Ext}^2_R(T, -) = 0$, i.e. $T \in \mathcal{P}_1$. Condition (T2) is clear by assumption.

Now we prove that $\mathrm{Pres}(T) = \mathrm{Gen}(T)$. Let $M \in \mathrm{Gen}(T)$. Then the canonical map $\varphi \in \mathrm{Hom}_R(T^{(\mathrm{Hom}_R(T,M))}, M)$ is surjective, so there is an exact sequence $0 \to K \to T^{(\mathrm{Hom}_R(T,M))} \xrightarrow{\varphi} M \to 0$. Applying $\mathrm{Hom}_R(T, -)$, we get

$$0 \to \mathrm{Hom}_R(T, K) \to \mathrm{Hom}_R(T, T^{(\mathrm{Hom}_R(T,M))}) \xrightarrow{\mathrm{Hom}_R(T,\varphi)} \mathrm{Hom}_R(T, M)$$
$$\to \mathrm{Ext}^1_R(T, K) \to \mathrm{Ext}^1_R(T, T^{(\mathrm{Hom}_R(T,M))}) = 0.$$

By definition, $\mathrm{Hom}_R(T, \varphi)$ is surjective, so $\mathrm{Ext}^1_R(T, K) = 0$ and $K \in \mathrm{Gen}(T)$.

It remains to verify condition (T3). By condition (T2) and Theorem 1.40, there is a special $T^{\perp}$–preenvelope, $\psi : R \hookrightarrow T_0$, of R with $T_1 = \mathrm{Coker}(\psi)$ isomorphic to a direct sum of copies of T. Since $R \in {}^{\perp}(T^{\perp})$, also $T_0 \in {}^{\perp}(T^{\perp})$. Since $\mathrm{Gen}(T) = \mathrm{Pres}(T)$, there are a cardinal λ and an exact sequence $0 \to K \to T^{(\lambda)} \to T_0 \to 0$ with $K \in T^{\perp}$. It follows that the sequence splits, and $T_0 \in \mathrm{Add}(T)$. So (T3) holds for $r = 1$. ■

In particular, if T is any 1–tilting module, then the 1–tilting class $T^\perp = \mathrm{Gen}(T)$ is a torsion class, called the *tilting torsion class* generated by T. The corresponding torsion–free class is $\operatorname{Ker}\operatorname{Hom}_R(T,-)$, so $(T^\perp, \operatorname{Ker}\operatorname{Hom}_R(T,-))$ is a (non–hereditary) torsion pair in Mod–R called the *tilting torsion pair* generated by T.

Now we will characterize tilting torsion classes among all torsion classes of modules in terms of approximations.

Theorem 3.29. *Let R be a ring and $\mathcal{T}$ be a class of modules. The following conditions are equivalent:*

(a) *$\mathcal{T}$ is a tilting torsion class.*

(b) *$\mathcal{T}$ is a special preenveloping torsion class.*

(c) *$\mathcal{T}$ is a torsion class such that R has a special $\mathcal{T}$–preenvelope.*

Proof. (a) implies (b): by Lemma 3.28, $\mathcal{T} = T^\perp$ for a 1–tilting module T. Since $\operatorname{proj\,dim} T \leq 1$, $\mathcal{T}$ is closed under homomorphic images. By Proposition 3.9, $\mathcal{T}$ is closed under direct sums. Now (b) follows by Theorem 1.40 (b).

(b) implies (c): this is trivial.

(c) implies (a): let $0 \to R \to T_0 \to T_1 \to 0$ be a special $\mathcal{T}$–preenvelope of R. We will prove that $T = T_0 \oplus T_0$ is a 1–tilting module such that $\mathrm{Gen}(T) = \mathcal{T}$.

Since $\mathcal{T}$ is a pretorsion class, we have $T \in \mathcal{T}$, and $\mathrm{Gen}(T) \subseteq \mathcal{T}$. Let $M \in T^\perp (= T_1^\perp)$. The pushout argument from the proof of Lemma 3.8 (b) (for $r = 1$) shows that $M \in \mathrm{Gen}(T)$. Finally, the $\mathcal{T}$–preenvelope of R is special, so $T_1 \in {}^\perp\mathcal{T}$, and $\mathcal{T} \subseteq ({}^\perp\mathcal{T})^\perp \subseteq T_1^\perp = T^\perp$.

This proves that $T^\perp = \mathrm{Gen}(T) = \mathcal{T}$, so T is 1–tilting by Lemma 3.28. ∎

The equivalence of parts (a) and (b) is in fact a particular case of Theorem 3.12 for $n = 1$. Indeed, it suffices to verify that ${}^\perp\mathcal{T} \subseteq \mathcal{P}_1$ in case $\mathcal{T}$ is a special preenveloping torsion class. But such $\mathcal{T}$ contains all homomorphic images of injective modules. If $M \in {}^\perp\mathcal{T}$, $N \in$ Mod–R and E is the injective hull of N, then $0 = \operatorname{Ext}^1_R(M, E/N) \to \operatorname{Ext}^2_R(M, N) \to \operatorname{Ext}^2_R(M, E) = 0$, and $\operatorname{Ext}^2_R(M, -) = 0$. So ${}^\perp\mathcal{T} \subseteq \mathcal{P}_1$ as required.

It is worthwhile to restate Corollary 3.13 and Theorem 3.27 in the particular setting of modules of projective dimension ≤ 1:

Corollary 3.30. *Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then the following assertions are equivalent:*

(a) *$\mathfrak{C}$ is a 1–tilting cotorsion pair.*

(b) *$\mathcal{A} \subseteq \mathcal{P}_1$, and $\mathcal{B}$ is closed under direct sums.*

Corollary 3.31. *Let R be a ring. Then 1–tilting torsion classes $\mathcal{C}$ correspond bijectively to the classes $\mathcal{S}$ such that $R \in \mathcal{S}$, $\mathcal{S} \subseteq \mathcal{P}_1^{<\omega}$ and $\mathcal{S}$ is closed under extensions and direct summands. The correspondence is given by the mutually inverse assignments $\mathcal{C} \mapsto ({}^\perp\mathcal{C})^{<\omega}$ and $\mathcal{S} \mapsto \mathcal{S}^\perp$.*

Proof. This follows immediately from Lemma 3.26 and Theorem 3.27. ∎

It turns out that in the particular setting of artin algebras R, torsion classes in mod–R are sufficient to classify all 1–tilting classes in Mod–R. Our presentation of this fact follows [57].

The tool making the classification possible is an infinitary version of a well–known formula by Auslander and Reiten. For its proof we refer to [34].

Lemma 3.32. *Let R be an artin algebra, D be the standard duality and $\tau = DTr$, $\tau^- = TrD$ be the Auslander–Reiten translations in* mod–R. *Let $M \in \mathcal{P}_1^{<\omega}$ and $N \in$ Mod–R. Then $D(\operatorname{Ext}^1_R(M,N)) \cong \operatorname{Hom}_R(N, \tau M)$.*

The following extends the characterization of tilting torsion classes generated by classical tilting modules over artin algebras by Assem and Hoshino (see [10, §VI.6]):

Theorem 3.33. *Let A be an artin algebra. Then there is a bijective correspondence between 1–tilting classes $\mathcal{C} \subseteq$ Mod–R and torsion classes $\mathcal{T} \subseteq$ mod–R such that $\mathcal{T}$ contains all finitely generated injective modules. The correspondence is given by the mutually inverse maps*

$$f : \mathcal{C} \mapsto \mathcal{C}^{<\omega}$$

and

$$g : \mathcal{T} \mapsto \operatorname{Ker}\operatorname{Hom}_R(-, \mathcal{F})$$

where $(\mathcal{T}, \mathcal{F})$ is a torsion pair in mod–R.

Proof. If $\mathcal{C}$ is a torsion class in Mod–R, then $f(\mathcal{C})$ is a torsion class in mod–R, since R is right noetherian (see Lemma 2.57 (a)). So f is well–defined.

Let $\mathcal{T}$ be a torsion class in mod–R containing $\mathcal{I}_0^{<\omega}$ with the corresponding torsion pair $(\mathcal{T}, \mathcal{F})$ in mod–R. Then $\mathcal{T}$ contains all finitely generated cosyzygies of all simple modules, hence ${}^\perp\mathcal{T}$ consists of modules of projective dimension ≤ 1. (Indeed, if S is a simple module with injective hull $E(S)$, consider the short exact sequence $0 \to S \to E(S) \to X \to 0$. For $M \in {}^\perp\mathcal{T}$, we have $0 = \operatorname{Ext}^1_R(M,X) \cong \operatorname{Ext}^2_R(M,S)$. Since $\operatorname{Ext}^2_R(M,S) = 0$ holds for all simple modules S, we get $\operatorname{proj\,dim} M \leq 1$ by Lemma 2.8.)

By Lemma 3.32 (a), for each $M \in$ mod–R, $M \in {}^\perp\mathcal{T}$, iff $\operatorname{Hom}_R(\mathcal{T}, \tau M) = 0$, iff $\tau M \in \mathcal{F}$. Put $\tau^-\mathcal{F} = \{M \in \text{mod–}R \mid \tau M \in \mathcal{F}\}$. Since $\mathcal{F}$ contains no non–zero injective modules, we have $\tau(\tau^- F) = F$ for each $F \in \mathcal{F}$. As $\tau^-\mathcal{F}$ consists of modules of projective dimension ≤ 1, Lemma 3.32 (a) yields $g(\mathcal{T}) = \operatorname{Ker}\operatorname{Hom}_R(-, \tau(\tau^-\mathcal{F})) = (\tau^-\mathcal{F})^\perp$, so $g(\mathcal{T})$ is a class of finite type, hence 1–tilting, in Mod–R, and g is well–defined.
Clearly $\mathcal{T} = \{M \in \text{mod–}R \mid \operatorname{Hom}_R(M,F) = 0 \text{ for all } F \in \mathcal{F}\} = fg(\mathcal{T})$.

Conversely, let $\mathcal{C}$ be a 1–tilting class in Mod–R. Let $\mathcal{T} = f(\mathcal{C})$, $(\mathcal{T}, \mathcal{F})$ be a torsion pair in mod–R and $\mathcal{D} = gf(\mathcal{C})$. Then $f(\mathcal{D}) = fgf(\mathcal{C}) = f(\mathcal{C})$, that is, the finitely generated modules in $\mathcal{C}$ and $\mathcal{D}$ coincide.

We claim that also the pure–injective modules in $\mathcal{C}$ and $\mathcal{D}$ coincide. To see this, let M be a module and $(f_i : M \to F_i \mid i \in I)$ a representative set (up to isomorphism) of all epimorphisms from M onto a finitely generated module. Then any homomorphism from M to a finitely generated module has a factorization through $f : M \to \prod_{i\in I} F_i$. Then the map f is a pure embedding. Since $\mathcal{C}$ is a torsion class in Mod–R, we infer that a pure–injective module M belongs to $\mathcal{C}$, iff M is a direct summand in a (possibly infinite) direct product of elements of $f(\mathcal{C})$, and similarly for $\mathcal{D}$. However, $f(\mathcal{C}) = f(\mathcal{D})$, and the claim follows.

Since $\mathcal{D} = (\tau^- \mathcal{F})^\perp$, the classes $\mathcal{C} = \mathcal{S}^\perp$ and $\mathcal{D}$ are of finite type, so they are definable by Theorem 3.15. In particular, a module belongs to $\mathcal{C}$, if and only if its pure–injective envelope does (see Lemma 1.38), and similarly for $\mathcal{D}$. It follows that $\mathcal{C} = \mathcal{D}$, that is, $\mathcal{C} = gf(\mathcal{C})$.

■

As an example, we will consider the correspondence from Theorem 3.33 in the particular setting of Examples 3.4 and 3.5.

Example 3.34. Let R be a connected tame hereditary algebra over a field k, P be a non–empty set of tubes, and T_P be the corresponding Ringel tilting module (see Example 3.4). Then the corresponding torsion class in mod–R is $T_P^\perp \cap \text{mod}–R$ which consists of all preinjective modules and all regular modules in the tubes not in P.

If R is a connected hereditary algebra of infinite representation type and T_P is the Lukas tilting module from Example 3.5 (b), then the corresponding torsion class in mod–R consists of all regular modules and all preinjective modules.

If R is a connected wild hereditary algebra and T_M is the Lukas divisible module from Example 3.5 (a), then the corresponding torsion class in mod–R is the class of all preinjective modules.

Notice that neither of the tilting modules T_P and T_M above is equivalent to a finitely generated tilting module.

As our second example, we consider the case of Dedekind domains.

Recall from Example 3.3 that given a Dedekind domain R and a set of maximal ideals $P \subseteq \operatorname{mspec} R$, R_P denotes the module $\pi^{-1}(\bigoplus_{p\in P} E(R/p))$, where $\pi : Q \to Q/R$ is the canonical projection. By 3.3, $T_P = R_P \oplus \bigoplus_{p\in P} E(R/p)$ is a 1–tilting module with the corresponding tilting class

$$T_P^\perp = (\bigoplus_{p\in P} R/p)^\perp = \{M \in \text{Mod–}R \mid Mp = M \text{ for all } p \in P\}.$$

Theorem 3.35. *Let R be a Dedekind domain.*

(a) *There is a bijective correspondence between the set of all tilting classes in* Mod–R *and all subsets of the maximal spectrum* $\operatorname{mspec} R$. *The correspondence is given by the mutually inverse assignments*

$$\mathcal{C} \mapsto P_\mathcal{C} = \{p \in \operatorname{mspec} R \mid Mp = M \textit{ for all } M \in \mathcal{C}\}$$

and

$$P \mapsto \mathcal{C}_P = \{M \in \text{Mod–}R \mid Mp = M \textit{ for all } p \in P\}.$$

(b) *The set of Bass tilting modules* $\{T_P \mid P \subseteq \operatorname{mspec} R\}$ *is a representative set* (*up to equivalence*) *of the class of all tilting modules.*

Proof. (a) Since R is a hereditary noetherian ring, by Theorem 3.27, tilting classes in Mod–R correspond bijectively to the resolving subcategories $\mathcal{S} \subseteq$ mod–R, that is, the subcategories $\mathcal{S}$ containing all finitely generated projective modules and closed under finite direct sums, direct summands (and submodules).

By Steinitz Theorem, each finitely generated R–module is a finite direct sum of a projective module and some modules of the form R/p^n where p are maximal ideals of R and $0 < n < \omega$. So resolving subcategories $\mathcal{S}$ correspond bijectively to subsets $P \subseteq \operatorname{mspec} R$ as follows: given P, $\mathcal{S}_P$ consists of all finitely generated modules whose q–primary components are zero for all $q \notin P$. The corresponding tilting class $\mathcal{T}_P = \mathcal{S}_P^{\perp} = \{M \in \text{Mod–}R \mid \operatorname{Ext}^1_R(R/p, M) = 0 \text{ for all } p \in P\} = \mathcal{C}_P$.

(b) We show that any tilting module T is equivalent to T_P for some $P \subseteq \operatorname{mspec} R$. By part (a), $T^{\perp} = \{M \in \text{Mod–}R \mid Mp = M \text{ for all } p \in P\}$ for a set of maximal ideals P. But the latter class equals $T_P^{\perp}$, so T is equivalent to T_P. Finally, if $P \neq P'$, then T_P is not equivalent to $T_{P'}$ (by part (a), or simply because $R/p \in T_{P'}^{\perp} \setminus T_P^{\perp}$ whenever $p \in P \setminus P'$). ∎

We finish this chapter by considering the dual case of cotilting (left R–) modules. We start with a definition:

Definition 3.36. A left R–module C is *cotilting* provided that

(C1) C has finite injective dimension.

(C2) $\operatorname{Ext}^i_R(C^\kappa, C) = 0$ for all $1 \leq i < \omega$ and all cardinals κ.

(C3) There is $r \geq 0$ and a long exact sequence $0 \to C_r \to \ldots \to C_1 \to C_0 \to W \to 0$, where $C_i \in \operatorname{Prod}(C)$ for all $i \leq r$ and W is an injective cogenerator for R–Mod.

If $n < \omega$ and C is a cotilting left R–module of injective dimension $\leq n$, then C is called *n–cotilting*. The class ${}^{\perp_\infty}C$ ($\subseteq R$–Mod) is the *n–cotilting class induced by C*. Clearly $({}^{\perp_\infty}C, ({}^{\perp_\infty}C)^{\perp})$ is a hereditary cotorsion pair in R–Mod, called the *n–cotilting cotorsion pair* induced by C.
If C and C' are cotilting left R–modules, then C' is *equivalent* to C provided that the induced cotilting classes coincide, that is, ${}^{\perp_\infty}C = {}^{\perp_\infty}C'$.

Clearly 0–cotilting modules coincide with injective cogenerators for R–Mod. Since each tilting module is of finite type, the duality $(-)^d$ yields an easy way of producing n–cotilting left R–modules from n–tilting (right R–) modules. (Recall that given a right R–module M, the dual left R–module, M^d, is defined by $M^d = \operatorname{Hom}_S(M, E)$, where E is an injective cogenerator for S–Mod and R is an S–algebra.)

Theorem 3.37. *Let R be a ring, $n \geq 0$ and T be an n–tilting module. Then the dual module T^d is an n–cotilting left R–module with the induced n–cotilting class*

$$\mathcal{C} = T^{\intercal_\infty} = \{M \in R\text{–}Mod \mid \operatorname{Tor}_i^R(T, M) = 0 \text{ for all } i \leq n\}.$$

Moreover, if $\mathcal{X} \subseteq \mathcal{P}_n^{<\omega}$ is such that $T^{\perp_\infty} = \mathcal{X}^{\perp_\infty}$, then $\mathcal{C} = \mathcal{X}^{\intercal_\infty}$.

Proof. Clearly, if $M \in \text{Mod–}R$ is projective, then M^d is an injective left R–module. Similarly, if M is a generator for Mod–R, then M^d is a cogenerator for R–Mod, and $M \in \text{Add}(T)$ implies $M^d \in \text{Prod}(T^d)$. This proves conditions (C1) and (C3) for $C = T^d$.

Let κ be a cardinal. Then for each $i \geq 1$, $\text{Ext}^i_R(C^\kappa, C) = 0$, iff $\text{Tor}^R_i(T, C^\kappa) = 0$. Let $\mathcal{S} = ({}^\perp(T^{\perp_\infty}))^{<\omega}$ ($\subseteq \mathcal{P}_n$). By Theorem 3.25, T is of finite type, so $\mathcal{S}^\perp = T^{\perp_\infty}$. Let $U = T \oplus \Omega^1(T) \oplus \cdots \oplus \Omega^{n-1}(T)$, the direct sum of the syzygies of T. Then $T \in {}^\perp(U^\perp) = {}^\perp(\mathcal{S}^\perp) \subseteq \varinjlim \mathcal{S}$ by Corollary 2.62.

Since the Tor–functor commutes with $\varinjlim$, in order to prove condition (C2) it suffices to show that $\text{Tor}^R_i(\mathcal{S}, C^\kappa) = 0$ (for each $i \geq 1$ and each cardinal κ). However, $\mathcal{S} \subseteq \text{mod–}R$ and $C^\kappa = (T^{(\kappa)})^d$, so the latter is equivalent to $\text{Ext}^i_R(\mathcal{S}, T^{(\kappa)}) = 0$. This holds since $\mathcal{S} \subseteq {}^\perp(T^{\perp_\infty})$ and $T^{(\kappa)} \in T^{\perp_\infty}$ by condition (T2) defining the tilting module T.

Now ${}^{\perp_\infty}C = T^{\intercal_\infty}$, which gives the first claim since T has projective dimension $\leq n$.

For the final claim, let $\mathcal{Q}$ be a set containing R and a representative set of elements of $\mathcal{X}$ and their syzygies. Then $\mathcal{Q}^\perp = \mathcal{X}^{\perp_\infty} = U^\perp$ and $\mathcal{Q}^\intercal = \mathcal{X}^{\intercal_\infty}$. By Corollary 1.41, $\mathcal{Q}$ consists of direct summands of $\{U, R\}$–filtered modules, and vice versa, U is a direct summand of a $\mathcal{Q}$–filtered module. By Corollary 1.32, $\mathcal{Q}^\intercal = U^\intercal = T^{\intercal_\infty}$, so $\mathcal{C} = \mathcal{X}^{\intercal_\infty}$. ∎

We pause to consider an example:

Example 3.38. Let R be a commutative 1–Iwanaga–Gorenstein ring (see Example 3.3). For $i = 0, 1$ let $P_i = \{p \in \text{spec}R \mid \text{ht}(p) = i\}$, and let $Q = \bigoplus_{p \in P_0} E(R/p)$.

As in Example 3.3, R has a minimal injective coresolution of the form

$$0 \to R \to \bigoplus_{q \in P_0} E(R/q) \xrightarrow{\pi} \bigoplus_{p \in P_1} E(R/p) \to 0,$$

and for each subset $P \subseteq P_1$ there is a Bass (1–) tilting module $T_P = R_P \oplus \bigoplus_{p \in P} E(R/p)$, where $R_P = \pi^{-1}(\bigoplus_{p \in P} E(R/p))$, and T_P generates the tilting class $T_P^\perp = \{M \mid \text{Ext}^1_R(E(R/p), M) = 0 \text{ for all } p \in P\}$.

Consider the injective cogenerator $E = \bigoplus_{p \in \text{mspec}\, R} E(R/p)$. By Theorem 3.37, $C_P = (T_P)^d = \text{Hom}_R(T_P, E)$ is a (1–) cotilting module, called the *Bass cotilting module*. Notice that $(\bigoplus_{p \in P} E(R/p))^d \cong \prod_{p \in P} J_p$, the product of the p–adic modules over $p \in P$. Since Q is a flat module and the sequence $0 \to R_P \to Q \to \bigoplus_{q \in P_1 \setminus P} E(R/q) \to 0$ is exact and its last term is injective, hence of flat dimension ≤ 1, we infer that R_P is flat, and hence $(R_P)^d$ is injective. So the corresponding cotilting class is

$$\mathcal{C}_P = {}^\perp C_P = \{M \in \text{Mod–}R \mid \text{Ext}^1_R(M, \prod_{p \in P} J_p) = 0\}$$

$$= \{M \in \text{Mod–}R \mid \text{Tor}^R_1(E(R/p), M) = 0 \quad \text{for all } p \in P\}.$$

There is a dual version of Lemma 3.8 for cotilting modules:

Lemma 3.39. *Let R be a ring, $n \geq 0$ and C be an n–cotilting left R–module. Denote by $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ the n–cotilting cotorsion pair induced by C.*

(a) $\mathcal{A} \subseteq \mathrm{Cogen}(C)$ *and* $\mathcal{B} \subseteq \mathcal{I}_n$.

(b) *Each of the short exact sequences forming the long exact sequence in (C3) is given by a special $\mathcal{A}$–precover of an element of $\mathcal{B}$. The length r in (C3) can be taken $\leq n$.*

(c) *The kernel of $\mathfrak{C}$ equals* $\mathrm{Prod}(C)$.

(d) *Each $M \in \mathcal{A} \cap \mathcal{I}_n$ has* $\mathrm{Prod}(T)$*–coresolution dimension $\leq n$.*

Proof. (a) Since $C \in \mathcal{I}_n$ and $({}^{\perp}\mathcal{I}_n, \mathcal{I}_n)$ is a cotorsion pair (see Theorem 2.7), we have $\mathcal{B} \subseteq \mathcal{I}_n$.

The rest of the proof is dual to the one for Lemma 3.8 (with the injective cogenerator W in R–Mod replacing the projective generator R). ■

By Lemma 3.39 (c), the kernel of the cotorsion pair $\mathfrak{C}$ equals $\mathrm{Prod}(C)$. The proof of Proposition 3.9 can be dualized to prove

Proposition 3.40. *Let R be a ring, $n \geq 0$, C be an n–cotilting left R–module and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ the n–cotilting cotorsion pair induced by C.*

(a) *$\mathcal{A}$ coincides with the class of all* $\mathrm{Prod}(C)$*–coresolved modules. In particular, $\mathcal{A}$ is closed under direct products.*

(b) *$\mathcal{B}$ coincides with the class of all* $\mathrm{Prod}(C)$*–resolved modules of* $\mathrm{Prod}(C)$*–resolution dimension $\leq n$.*

As in Corollary 3.10, we can replace $\mathrm{Prod}(C)$–(co)resolutions in Proposition 3.40 by $\mathcal{Q}$–(co)resolutions, where $\mathcal{Q} = \{C^{\kappa} \mid \kappa \geq 0\}$; n–cotilting left R–modules can then be characterized by the dual property of ${}^{\perp_\infty}C = \mathrm{Cog}_n(C)$.

Moreover, dually to Lemma 3.11, we obtain that if C_1 and C_2 are cotilting left R–modules, then C_1 is equivalent to C_2, iff $\mathrm{Prod}(C_1) = \mathrm{Prod}(C_2)$, iff $\mathrm{Prod}(C_1) \subseteq \mathrm{Prod}(C_2)$.

The crucial property of cotilting modules making the dualization of the theory of tilting modules possible is their pure–injectivity. It was proved in many steps: first for abelian groups [46], and then for modules over Dedekind and Prüfer domains [39], [22], and for 1–cotilting modules over arbitrary rings [17]. The general case of n–cotilting modules was first settled for countable rings in [22]; the case of arbitrary rings is due to Šťovíček, [70].

Theorem 3.41. *Let R be a ring, $n \geq 0$ and C be an n–cotilting left R–module. Denote by $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ the n–cotilting cotorsion pair induced by C.*

Then C is pure–injective, $\mathcal{A}$ is definable, and $\mathfrak{C}$ is a perfect cotorsion pair.

Proof. By Lemma 3.39 (a), $\mathcal{B} \subseteq \mathcal{I}_n$. By Proposition 3.40 (a), $\mathcal{A}$ is closed under direct products. Since $\mathfrak{C}$ is a hereditary cotorsion pair, Theorem 2.35 applies and yields that $\mathcal{A}$ is definable and $\mathfrak{C}$ is perfect.

In order to prove that C is pure–injective, it suffices to show that for each cardinal κ, $\mathrm{Ext}^1_R(C^{\kappa}/C^{(\kappa)}, C) = 0$. Then the summation map $\Sigma : C^{(\kappa)} \to C$ extends to C^{κ}, so C is pure–injective by [56, §7].

Clearly the module $C^{(\kappa)}$ is a direct limit of the direct system consisting of the modules $C^{(F)}$, where F runs over all finite subsets of κ. Then $C^{\kappa}/C^{(\kappa)} =$

$\varinjlim_F C^\kappa/C^{(F)}$. Since $C^\kappa/C^{(F)} \cong C^{\kappa\setminus F} \in \mathcal{A}$ by condition (C2), and $\mathcal{A}$ is closed under direct limits, we conclude that $C^\kappa/C^{(\kappa)} \in \mathcal{A}$. ∎

A bimodule ${}_RC_S$ is an *n–cotilting bimodule* provided that $C \in R$–Mod and $C \in$ Mod–S are n–cotilting modules and ${}_RC_S$ is faithfully balanced. For example, *Morita bimodules* (i.e., faithfully balanced bimodules which are injective cogenerators on either side) are exactly the 0–cotilting bimodules.

Since cotilting modules are pure–injective, their endomorphism rings S have rather strong properties: $S/\mathrm{Rad}(S)$ is von Neumann regular, right self–injective, and idempotents lift modulo $\mathrm{Rad}(S)$. This follows from the analogous well–known properties of endomorphism rings of injective objects in Grothendieck categories (cf. the proof of Proposition 1.10). These strong properties explain why cotilting bimodules occur rarely as compared to tilting bimodules.

If ${}_RC_S$ is a Morita bimodule, then C induces a Morita duality between R–mod and mod–S, [2, §24]. Similarly, each 1–cotilting bimodule induces a "generalized Morita duality". We refer to [31] for the role of cotilting bimodules in the duality theory for module categories.

Now we can characterize n–cotilting classes and n–cotilting cotorsion pairs by their closure properties:

Theorem 3.42. *Let R be a ring, $n < \omega$ and $\mathcal{C}$ be a class of left R–modules. Then the following assertions are equivalent:*

(a) *$\mathcal{C}$ is n–cotilting.*

(b) *$\mathcal{C}$ is resolving, covering, closed under direct products and direct summands, and $\mathcal{C}^\perp \subseteq \mathcal{I}_n$.*

Proof. (a) implies (b): this is an immediate consequence of Lemma 3.39 (a) and Theorem 3.41.

(b) implies (a): this is proved dually to the implication (b) implies (a) in Theorem 3.12. ∎

Corollary 3.43. *Let $n < \omega$. Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair in R–Mod. Then the following assertions are equivalent:*

(a) *$\mathfrak{C}$ is an n–cotilting cotorsion pair.*

(b) *$\mathfrak{C}$ is a hereditary (and perfect) cotorsion pair such that $\mathcal{B} \subseteq \mathcal{I}_n$ and $\mathcal{A}$ is closed under direct products.*

Proof. (a) implies (b): by Theorems 3.41 and 3.42 for $\mathcal{C} = \mathcal{A}$.

(b) implies (a): in view of Theorem 3.42, we only have to prove that perfectness of $\mathfrak{C}$ follows from the other assumptions on $\mathfrak{C}$. However, this holds by Theorem 2.35 (b). ∎

By Theorem 3.37, the dual module of any tilting module is cotilting. Similarly as all tilting modules and classes are of finite type, the duals of tilting modules and classes are exactly the cotilting modules and classes of cofinite type, defined as follows:

Definition 3.44. Let R be a ring.
Let $\mathcal{C}$ be a class of left R–modules. Then $\mathcal{C}$ is of *cofinite type* provided there exist $n < \omega$ and $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ such that $\mathcal{C} = \mathcal{S}^{\intercal_\infty}$.
Let C be a left R–module. Then C is of *cofinite type* provided that the class ${}^{\perp_\infty}C$ is of cofinite type.

Let $\mathcal{C}$ be a class of cofinite type and $\mathcal{A} = {}^{\intercal}\mathcal{C}\,(= {}^{\intercal_\infty}\mathcal{C})$. Then clearly $\mathcal{C} = (\mathcal{A}^{<\omega})^{\intercal_\infty}$, so $\mathcal{S} = \mathcal{A}^{<\omega}$ is the largest possible choice for $\mathcal{S}$ in the Definition 3.44.

Any class of cofinite type is a cotilting class:

Proposition 3.45. *Let R be a ring and $\mathcal{C}$ be a class of left R–modules of cofinite type. Then $\mathcal{C}$ is cotilting (and definable).*

Proof. By assumption, there are $n < \omega$ and $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ such that $\mathcal{C} = \mathcal{S}^{\intercal_\infty}$. Let $\mathcal{S}^d = \{S^d \mid S \in \mathcal{S}\}$. Then $\mathcal{C} = {}^{\perp_\infty}(\mathcal{S}^d)$ by Lemma 1.31 (b), so $\mathcal{C}$ is resolving, and it is a covering class by Corollary 1.51. Since $\mathcal{S} \subseteq \text{mod–}R$, the functor $\text{Tor}_1^R(S, -)$ commutes with direct products, whenever $S \in \mathcal{S}$ or $S \in \text{mod–}R$ is a syzygy of a module in $\mathcal{S}$ (see e.g. [40, §3.2]). So $\mathcal{C}$ is closed under direct products. Since $\mathcal{S}^d \subseteq \mathcal{I}_n$, also $\mathcal{C}^\perp \subseteq \mathcal{I}_n$. By Corollary 3.42, we infer that $\mathcal{C}$ is an n–cotilting class.

Finally, each cotilting class is definable by Theorem 3.41. ∎

Notice that if $\mathcal{C}$ is of cofinite type, then, by Lemma 1.31, the least n such that the class $\mathcal{C} = {}^{\perp_\infty}(\mathcal{S}^d)$ is n–cotilting coincides with the least n such that $\mathcal{S} \subseteq \mathcal{F}_n$. However, $\mathcal{S} \subseteq \text{mod–}R$, and finitely presented flat modules are projective, so this is exactly the least n such that $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$.

The essential difference between the tilting and cotilting setting is that the converse of Proposition 3.45 does not hold in general. In Example 3.55 below, we will construct a 1–cotilting class over a valuation domain which is not of cofinite type. So in general there exist more cotilting modules than just duals of the tilting ones. However, for many classes of rings, all cotilting modules are of cofinite type, so this surprising phenomenon does not occur.

Theorem 3.46. *Let R be a ring and $n < \omega$.*

(a) *Let C be an n–cotilting left R–module. Then C is of cofinite type, if and only if there is an n–tilting module T_C such that C is equivalent to T_C^d.*

(b) *If C and C' are n–cotilting modules of cofinite type, then C' is equivalent to C, if and only if the n–tilting modules T_C and $T_{C'}$ are equivalent. In particular, T_C is uniquely determined by C up to equivalence of tilting modules.*

Proof. (a) Assume that C is of cofinite type. By the remark after Definition 3.44, there is $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ such that ${}^{\perp_\infty}C = \mathcal{S}^{\intercal_\infty}$ and we can w.l.o.g. assume that $\mathcal{S} = {}^{\intercal}(\mathcal{S}^{\intercal_\infty}) \cap \text{mod–}R$ (so in particular, $\mathcal{S}$ a resolving subcategory of mod–R). Since the class $\mathcal{S}^\perp$ is of finite type (see Definition 3.14), there is an (n–) tilting module T_C such that $T_C^{\perp_\infty} = \mathcal{S}^\perp$. By Theorem 3.37, T_C^d is an n–cotilting left R–module inducing the cotilting class $T_C^{\intercal_\infty} = \mathcal{S}^{\intercal_\infty} = {}^{\perp_\infty}C$, so T_C^d is equivalent to C.

Conversely, assume that C is equivalent to T^d for an n–tilting module T. Since T is of finite type, there is $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ such that $T^{\perp_\infty} = \mathcal{S}^{\perp_\infty}$, so ${}^{\perp_\infty}C = T_C^{\intercal_\infty} = \mathcal{S}^{\intercal_\infty}$ by Theorem 3.37, hence C is of cofinite type.
(b) Assume that T and T' are tilting modules such that $C = T^d$ is equivalent to $C' = (T')^d$. Then $(T')^{\intercal_\infty} = T^{\intercal_\infty}$ by Lemma 1.31. Let $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}', \mathcal{B}')$ be the tilting cotorsion pairs induced by T and T'. Let $\mathcal{S} = \mathcal{A}^{<\omega}$ and $\mathcal{S}' = (\mathcal{A}')^{<\omega}$. By Theorem 2.61, $\mathcal{S} = (\varinjlim \mathcal{S})^{<\omega} = {}^{\intercal}(\mathcal{S}^{\intercal}) \cap \text{mod–}R = {}^{\intercal}(T^{\intercal_\infty}) \cap \text{mod–}R$, and similarly for $\mathcal{S}'$ and T', so $\mathcal{S} = \mathcal{S}'$. By Theorem 3.25, $\mathcal{S}^{\perp} = \mathcal{B}$, and similarly $(\mathcal{S}')^{\perp} = \mathcal{B}'$ so $\mathcal{B} = \mathcal{B}'$, that is, T and T' are equivalent tilting modules.
Conversely, if T_C and $T_{C'}$ are equivalent, then $T_{C'} \in \text{Add}(T_C)$, hence $T_{C'}^d \in \text{Prod}(T_C^d)$, so $T_{C'}^d$ is equivalent to T_C^d. ∎

In particular, if $\mathcal{U} = \{U_i \mid i \in I\}$ is a representative set of all tilting modules up to equivalence, then $\mathcal{U}^d = \{U_i^d \mid i \in I\}$ is a representative set of all cotilting modules of cofinite type up to equivalence.

The classes of finite type correspond to resolving subcategories in mod–R. In view of Theorem 3.46, it is not surprising that there is a similar correspondence for the classes of cofinite type:

Theorem 3.47. *Let R be a ring and $n < \omega$. There is a bijective correspondence between n–cotilting classes of cofinite type in R–Mod and resolving subcategories $\mathcal{S}$ of* mod–R *such that $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$. The correspondence is given by the mutually inverse assignments $\mathcal{C} \mapsto ({}^{\intercal}\mathcal{C})^{<\omega}$ and $\mathcal{S} \mapsto \mathcal{S}^{\intercal}$.*

Proof. Let $\mathcal{C}$ be an n–cotilting class of cofinite type in R–Mod. So there is a class $\mathcal{T} \subseteq \mathcal{P}_n^{<\omega}$ such that $\mathcal{C} = \mathcal{T}^{\intercal_\infty}$. Then clearly $({}^{\intercal}\mathcal{C})^{<\omega}$ is a resolving subcategory of mod–R.
Conversely, let $\mathcal{S}$ be a resolving subcategory of mod–R such that $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$. Then $\mathcal{C} = \mathcal{S}^{\intercal}$ is a class of cofinite type, so $\mathcal{C}$ is n–cotilting by Proposition 3.45.
Let $\mathcal{C}$ be an n–cotilting class of cofinite type in R–Mod, so $\mathcal{C} = \mathcal{S}^{\intercal_\infty}$ for a class $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$. W.l.o.g., $\mathcal{S} = ({}^{\intercal}\mathcal{C})^{<\omega}$, and hence $\mathcal{S}^{\intercal} = \mathcal{S}^{\intercal_\infty} = \mathcal{C}$.
Let $\mathcal{S}$ be a resolving subcategory of mod–R such that $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$. By Theorem 2.61, ${}^{\intercal}(\mathcal{S}^{\intercal}) = \varinjlim \mathcal{S}$. By Lemma 2.55, $\mathcal{S}$ coincides with the class of all finitely presented modules in $\varinjlim \mathcal{S}$. So $\mathcal{S} = (\varinjlim \mathcal{S})^{<\omega} = ({}^{\intercal}(\mathcal{S}^{\intercal}))^{<\omega}$. ∎

Now, we turn to the particular case of 1–cotilting modules. They can be characterized in terms of the classes they cogenerate as follows:

Lemma 3.48. *Let R be a ring. A module C is 1–cotilting, iff* $\text{Cogen}(C) = {}^{\perp}C$.

Proof. Dual to the proof of Lemma 3.28. ∎

It follows that if C is 1–cotilting, then $\text{Cogen}(C)$ is a torsion–free class of modules, called the *cotilting torsion–free class* cogenerated by C.
We will characterize cotilting torsion–free classes among all torsion–free classes of modules in terms of approximations:

Theorem 3.49. *Let R be a ring and $\mathcal{F}$ be a pretorsion–free class of left R–modules. Let W be an injective cogenerator for R–Mod. Then the following are equivalent:*

(a) *$\mathcal{F}$ is a cotilting torsion–free class.*

(b) *$\mathcal{F}$ is a covering class.*

(c) *$\mathcal{F}$ is a special precovering class.*

(d) *W has a special $\mathcal{F}$–precover.*

Proof. (a) implies (b) by Theorem 3.42, (b) implies (c) by Lemma 1.12 (b), and (c) trivially implies (d).

(d) implies (a): let $0 \to C_1 \to C_2 \to W \to 0$ be a special $\mathcal{F}$–precover of W. A dual proof to that of Theorem 3.29 shows that $C = C_1 \oplus C_2$ is a cotilting module such that $\mathrm{Cogen}(C) = \mathcal{F}$. ∎

Now we can easily show that in the particular case of left noetherian rings, 1–cotilting torsion–free classes are completely determined by their finitely generated modules. This result goes back to Buan and Krause [30].

Theorem 3.50. *Let R be a left noetherian ring. Then there is a bijective correspondence between the 1–cotilting classes $\mathcal{C}$ in R–Mod and the torsion–free classes $\mathcal{E}$ in R–mod containing R. The correspondence is given by the mutually inverse assignments*

$$\mathcal{C} \mapsto \mathcal{C} \cap R\text{–}mod \qquad \text{and} \qquad \mathcal{E} \mapsto \varinjlim \mathcal{E}.$$

Proof. If $\mathcal{C}$ is 1–cotilting then $\mathcal{C}$ is a torsion–free class in R–Mod containing R. By (the left–hand version of) Lemma 2.57 (a), $\mathcal{C} \cap R$–mod is a torsion–free class in R–mod.

Conversely, given $\mathcal{E}$ as in the claim, let $\mathcal{C} = \varinjlim \mathcal{E}$. By Lemma 2.57 (b), $\mathcal{C}$ is a torsion–free class in R–Mod. Since $R \in \mathcal{E}$, we have $\mathcal{C} = ({}^{\intercal}\mathcal{E})^{\intercal}$ by (a left R-module version of) Theorem 2.61. So $\mathcal{C}$ is a covering class by Theorem 1.54. By Theorem 3.49, $\mathcal{C}$ is 1–cotilting.

Now $\mathcal{E} = \varinjlim \mathcal{E} \cap R$–mod by Lemma 2.55. Conversely, given a 1–cotilting class $\mathcal{C}$ in R–Mod, each $M \in \mathcal{C}$ is a directed union of the system of all its finitely generated submodules, $\{M_i \mid i \in I\}$. Since $\mathcal{C}$ is 1–cotilting, $M_i \in \mathcal{C}$ for each $i \in I$. So $\mathcal{C} = \varinjlim(\mathcal{C} \cap R\text{–mod})$, and the assignments are mutually inverse. ∎

There is a general criterion for a 1–cotilting class to be of cofinite type:

Proposition 3.51. *Let R be a ring and $\mathcal{C}$ be a class of left R–modules. Then $\mathcal{C}$ is 1–cotilting of cofinite type, if and only if there is a module $M \in \mathcal{P}_1$ such that $\mathcal{C} = M^{\intercal}$.*

Proof. For the only–if–part, consider $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ such that $\mathcal{C} = \mathcal{S}^{\intercal_\infty}$. Let M be the direct sum of a representative set of all modules in $\mathcal{S}$. Then $\mathcal{C} = M^{\intercal_\infty} = {}^{\perp_\infty}M^d$. Since $\mathcal{C}$ is 1–cotilting, $\mathcal{C}^{\perp} \subseteq \mathcal{I}_1$, so $M^d \in \mathcal{I}_1$, hence $M \in \mathcal{F}_1$ by Lemma 1.31. So $\mathcal{S} \subseteq \mathcal{F}_1 \cap \text{mod–}R = \mathcal{P}_1^{<\omega}$ (since finitely presented flat modules are projective). So $M \in \mathcal{P}_1$, and $\mathcal{C} = M^{\intercal}$.

For the if–part, we consider the cotorsion pair $(\mathcal{A}, \mathcal{B})$ generated by M. Since $\mathcal{A} \subseteq \mathcal{P}_1$, the claim will follow once we prove that $M^{\intercal} = (\mathcal{A}^{<\omega})^{\intercal}$. If $N \in M^{\intercal}$, then $N^d \in \mathcal{B}$, so $N \in \mathcal{A}^{\intercal} \subseteq (\mathcal{A}^{<\omega})^{\intercal}$. Conversely, by Theorem 2.61 and Lemma 2.63, $M \in \varinjlim \mathcal{A}^{<\omega} = {}^{\intercal}((\mathcal{A}^{<\omega})^{\intercal})$, so $M^{\intercal} \supseteq (\mathcal{A}^{<\omega})^{\intercal}$. ∎

Now we can prove that in many cases, all 1–cotilting classes are of cofinite type:

Theorem 3.52. *Let R be a left noetherian ring such that $\mathcal{F}_1 = \mathcal{P}_1$ (this occurs when R is right hereditary, or right perfect, or 1–Iwanaga–Gorenstein). Then all 1–cotilting classes are of cofinite type, that is, all 1–cotilting left R–modules are equivalent to duals of 1–tilting (right R–) modules.*

Proof. Let $\mathcal{C} \subseteq R$–Mod be a 1–cotilting class. By a version of Theorem 2.61 for left R–modules, and by Theorem 3.50, $\mathcal{C} = \mathcal{D}^{\intercal}$, where $\mathcal{D} = {}^{\intercal}(\mathcal{C}^{<\omega})$. Since $\mathcal{C}$ is closed under submodules, we have $\mathcal{D} \subseteq \mathcal{F}_1$.

By Lemma 1.52, each module $D \in \mathcal{D}$ is $\mathcal{D}^{\leq\kappa}$–filtered, where $\kappa = |R| + \aleph_0$. Let M be the direct sum of a representative set of all modules in $\mathcal{D}^{\leq\kappa}$. By Corollary 1.32 $\mathcal{C} = M^{\intercal}$.

Finally, since $\mathcal{D} \subseteq \mathcal{F}_1 = \mathcal{P}_1$ by assumption, we conclude that $\mathcal{C}$ is of cofinite type by Proposition 3.51. ∎

In particular, if R is left artinian or 1–Iwanaga–Gorenstein, then we can describe 1–cotilting classes either by means of the torsion–free classes in R–mod containing R (as in Theorem 3.50), or by means of the subcategories $\mathcal{S} \subseteq$ mod–R closed under extensions and direct summands and satisfying $\mathcal{P}_0^{<\omega} \subseteq \mathcal{S} \subseteq \mathcal{P}_1^{<\omega}$ (as in Corollary 3.31).

As an application we consider in more detail the case of Dedekind domains. Given a Dedekind domain R and a set of maximal ideals $P \subseteq \operatorname{mspec} R$, we define a module Q_P by

$$Q_P = Q \oplus \bigoplus_{q \in \operatorname{mspec} R \setminus P} E(R/q) \oplus \prod_{p \in P} J_p.$$

Given an ideal I of R, we will call a module $M \in R$–Mod *I–torsion–free* provided that $\operatorname{Tor}_1^R(R/I, M) = 0$. Denote by $\mathcal{C}_P$ the class of all left R–modules that are p–torsion–free for all $p \in P$, that is,

$$\mathcal{C}_P = \{M \in R\text{–Mod} \mid \operatorname{Tor}_1^R(R/p, M) = 0 \text{ for all } p \in P\}.$$

Theorem 3.53. *Let R be a Dedekind domain.*

(a) *Let $\mathcal{C}$ be a class of modules. Then $\mathcal{C}$ is cotilting, iff there is a set of maximal ideals, P, such that $\mathcal{C} = \mathcal{C}_P$.*

(b) *The set $\{Q_P \mid P \subseteq \operatorname{mspec} R\}$ is a representative set (up to equivalence) of the class of all cotilting modules.*

Proof. (a) Clearly, given $P \subseteq \operatorname{mspec} R$, $\mathcal{C}_P$ is a class of cofinite type, hence a cotilting one by Proposition 3.45.

Conversely, if $\mathcal{C}$ is cotilting, then $\mathcal{C}$ is of cofinite type by Theorem 3.52, so by Theorems 3.35 and 3.46, there is a subset $P \subseteq \operatorname{mspec} R$ such that $\mathcal{C} = T_P^{\intercal}$. By Example 3.38, $\mathcal{C} = \{M \in \text{Mod–}R \mid \operatorname{Tor}_1^R(E(R/p), M) = 0 \ \forall p \in P\}$. Since R is hereditary and $E(R/p)$ is $\{R/p\}$–filtered for each $p \in \operatorname{mspec} R$, the condition $\operatorname{Tor}_1^R(E(R/p), M) = 0$ is equivalent to $\operatorname{Tor}_1^R(R/p, M) = 0$, for each $p \in P$, and the claim follows.

(b) In view of Theorems 3.35, 3.46 and 3.52, it suffices to prove that for each $P \subseteq \operatorname{mspec} R$, Q_P is a cotilting module equivalent to the cotilting module C_P defined in Example 3.38.

Condition (C1) for Q_P is clear. For (C2), since $Q \oplus J_P$ is flat and J_P is pure–injective, where $J_P = \prod_{p \in P} J_p$, it suffices to prove that $\operatorname{Ext}^1_R(I_\kappa, J_P) = 0$, where $I_\kappa = (\bigoplus_{q \in \operatorname{mspec} R \setminus P} E(R/q))^\kappa$, for each κ. However, I_κ is an injective module, and it is easy to see that I_κ has no indecomposable direct summands isomorphic to $E(R/p)$ for $p \in P$. Since J_p is q–divisible for all $p \in P$ and $q \in \operatorname{mspec} R \setminus P$, we infer that $\operatorname{Ext}^1_R(I_\kappa, J_P) = 0$. For condition (C3), consider the exact sequence $0 \to J_P \to E(J_P) \to E(J_P)/J_P \to 0$. Then $E(J_P) \cong Q^{(\lambda)}$ and $E(J_P)/J_P \cong \bigoplus_{p \in P} E(R/p)^{(\alpha_p)}$ for some cardinals $\lambda > 0$ and $\alpha_p > 0$ ($p \in P$). Let $W = E(J_P)/J_P \oplus \bigoplus_{q \in \operatorname{mspec} R \setminus P} E(R/q)$. Then W is an injective cogenerator for Mod–R, and the exact sequence $0 \to J_P \to E(J_P) \oplus \bigoplus_{q \in \operatorname{mspec} R \setminus P} E(R/q) \to W \to 0$ proves (C3).

Finally, ${}^\perp Q_P = {}^\perp C_P = {}^\perp J_P$, so Q_P and C_P are equivalent. ∎

In fact, the cotilting torsion–free classes over any Dedekind domain correspond bijectively to Tor–pairs:

Theorem 3.54. *Let R be a Dedekind domain. Let $\mathcal{C} \subseteq$ Mod–R. The following conditions are equivalent:*

(a) *$(\mathcal{C}, \mathcal{C}^\intercal)$ is a Tor–pair.*

(b) *$(\mathcal{C}, \mathcal{C}^\perp)$ is a cotorsion pair such that $\mathcal{C} \supseteq \mathcal{FL}$.*

(c) *There is a subset $P \subseteq \operatorname{mspec} R$ such that $\mathcal{C}$ is the class of all modules which are p–torsion–free for all $p \in P$.*

(d) *$\mathcal{C}$ is a cotilting torsion–free class.*

For a proof of Theorem 3.54, we refer to [46].

We finish this chapter by an example of a cotilting class that is not of cofinite type due to Bazzoni:

Example 3.55. Let R be a maximal valuation domain with a non–principal maximal ideal p (see [44, XIII.5]). Then R is pure–injective, so the class $\mathcal{W}_1$ of all Whitehead modules is a covering class by Theorem 1.48. Since R has injective dimension ≤ 1 by Theorem 2.45, $\mathcal{W}_1$ is closed under submodules and extensions. In order to prove that $\mathcal{W}_1$ is a cotilting torsion–free class, it suffices to show that $M \in \mathcal{W}_1$, iff p annihilates the torsion part $t(M)$ of M (then if $0 \neq x$ is an element in a direct product of Whitehead modules and $\operatorname{Ann}(x) \neq 0$, then $\operatorname{Ann}(x) = p$, so $\mathcal{W}_1$ is closed under direct products, and Theorem 3.49 applies).

First, since torsion–free modules are flat and R is pure–injective (see Theorem 2.45), $M \in \mathcal{W}_1$, iff $t(M) \in \mathcal{W}_1$.

Assume that $pt(M) = 0$. Then $t(M) \cong (R/p)^{(\alpha)}$, so it suffices to prove that $\operatorname{Ext}^1_R(R/p, R) = 0$. Note that $\operatorname{Ext}^1_R(R/p, R) \cong (Q/R)[p]/((Q[p] + R)/R)$, where $M[p]$ denotes the set of all elements $x \in M$ annihilated by p. (This follows by applying $\operatorname{Hom}_R(R/p, -)$ to the exact sequence $0 \to R \to Q \to Q/R \to 0$, and identifying $\operatorname{Hom}_R(R/p, N)$ with $N[p]$ for $N = Q$ and $N = Q/R$.) So $\operatorname{Ext}^1_R(R/p, R) = 0$, iff $(Q/R)[p] = 0$. Let $(R : p) = \{x \in Q \mid x.p \subseteq R\}$. Since p

is non–principal, $(R:p)=R$, so $(Q/R)[p]=(R:p)/R=0$. This proves that $pt(M)=0$ implies $M\in\mathcal{W}_1$.

Conversely, assume that M is a torsion Whitehead module and consider $0\neq x\in M$. Then $\operatorname{Ext}^1_R(Rx,R)=0$. Let $I=\operatorname{Ann}x$. Then as above $0=\operatorname{Ext}^1_R(R/I,R)\cong(Q/R)[I]/((Q[I]+R)/R)$, so $(Q/R)[I]=(R:I)/R=0$, and $(R:I)=R$. Then I is not principal, so $I(R:I)=I^\sharp$ where $I^\sharp$ is the prime ideal associated with I. But then $I=I^\sharp$, and since $(R:J)=R_{(J)}$ for every prime ideal J, we conclude that $I=p$. This proves that $M\in\mathcal{W}_1$ implies $pt(M)=0$, and finishes the proof that $\mathcal{W}_1$ is a cotilting torsion–free class.

Finally, we show that $\mathcal{W}_1$ is not of cofinite type. Assume there is $\mathcal{S}\subseteq$ mod–$R=\mathcal{P}_1^{<\omega}$ such that $\mathcal{W}_1=\mathcal{S}^\intercal$. Since R is a valuation domain, finitely presented modules coincide with direct sums of cyclically presented modules. So there is a set of non–zero elements $\{r_\alpha\mid\alpha<\kappa\}\subseteq p$ such that $\mathcal{W}_1=\{M\mid\operatorname{Tor}^R_1(R/r_\alpha R,M)=0\ \forall\alpha<\kappa\}$. However, $\operatorname{Tor}^R_1(R/r_\alpha R,M)\cong M[r_\alpha]$, so in particular $\operatorname{Tor}^R_1(R/r_\alpha R,R/p)\cong R/p\neq 0$. However, $R/p\in\mathcal{W}_1$, a contradiction.

4 Application 1: The structure of Matlis localizations

In this section we will employ infinitely generated 1–tilting modules in developing structure theory of *Matlis localizations*, that is, the localizations of commutative rings R at multiplicative sets S consisting of non–zero–divisors such that $\operatorname{proj}\dim S^{-1}R\leq 1$. As an application, we will then consider the existence of minimal versions of tilting approximations. Our approach is based on [5] and [72].

First, we will need a number of preliminary definitions and results. We start with the ones related to the spectrum, $\operatorname{spec}R$:

Let R be a commutative ring, and S a multiplicative subset of R. We write $V(S)=\{P\in\operatorname{spec}R\mid P\cap S=\emptyset\}$. Recall that $V(S)$ is canonically isomorphic to $\operatorname{spec}S^{-1}R$. Clearly, if $S\subseteq S'$, then $V(S')\subseteq V(S)$.

Denote by Σ the multiplicative subset consisting of all non–zero–divisors in R. Let S be a multiplicative subset of R. If s and s' are elements of R such that $ss'\in S$, then s and s' are invertible in the localization $S^{-1}R$. Hence $S^{-1}R=(S')^{-1}R$, where $S'=\{t\in R|s=tt' \text{ for some } s\in S \text{ and } t'\in R\}$ is a saturated multiplicative subset containing S. S' is called the *saturation* of S.

If S is a multiplicative subset of R, and I is an ideal of R such that $I\cap S=\emptyset$, then it is well–known that the set $\mathcal{C}=\{J\leq R\mid I\subseteq J \text{ and } J\cap S=\emptyset\}$ has maximal elements and that any such maximal element is a prime ideal of R.

Let now $S\subseteq\Sigma$ be a saturated multiplicative subset. Then any $x\in R\setminus S$ satisfies $xR\cap S=\emptyset$ and thus is contained in a prime ideal from $V(S)$, so $S=R\setminus\bigcup_{P\in V(S)}P$.

Similarly, if S is a multiplicative subset of Σ, then its saturation is easily seen to coincide with $S'=R\setminus\bigcup_{P\in V(S)}P$.

Observe that Σ is an example of a saturated multiplicative subset of R. Hence $R\setminus\Sigma$ is a union of prime ideals of R, and it can be proved that the minimal primes of R are in this union.

More generally, let M be a non–zero R–module. The set of elements in R such that multiplication by them induces an injective endomorphism of M is a saturated multiplicative subset of R.

Dually, the set of elements in R such that multiplication by them is a surjective endomorphism of M is also a saturated multiplicative subset of R.

We now present some results concerning the module $S^{-1}R/R$. In particular, we study its endomorphism ring and show that the direct sum decompositions of $S^{-1}R/R$ have quite nice properties.

Proposition 4.1. *Let R be a commutative ring. Let S be a multiplicative subset of Σ. Then R is canonically a subring of $S^{-1}R$ and the following holds true:*

(a) *If $f \in \mathrm{End}_R(S^{-1}R/R)$ and $x \in S^{-1}R/R$, then $f(x) \in xR$. In particular $\mathrm{End}_R(S^{-1}R/R)$ is a commutative ring.*

(b) *If $S^{-1}R/R = \bigoplus_{i\in I} A_i$ and $B \subseteq S^{-1}R/R$, then $B = \bigoplus_{i\in I}(B \cap A_i)$.*

(c) *If $S^{-1}R/R = A \oplus B = A \oplus B'$, then $B = B'$. Moreover, if A and A' are direct summands in $S^{-1}R/R$, then also $A \cap A'$ and $A + A'$ are direct summands in $S^{-1}R/R$.*

(d) *Assume $S^{-1}R/R = \bigoplus_{i\in I} A_i$. If $i \neq j$, then $\mathrm{Hom}_R(A_i, A_j) = 0$.*

Proof. (a) Let $f \in \mathrm{End}_R(S^{-1}R/R)$. Let $x \in S^{-1}R/R$, and let $s \in S$. Then $sx = 0$, if and only if $x \in (\frac{1}{s} + R)R$. Hence $f(\frac{1}{s} + R) \in (\frac{1}{s} + R)R$ for any $s \in S$, and then $f(x) \in xR$ for any $x \in S^{-1}R/R$.

(b) For any $i \in I$, let $\pi_i\colon S^{-1}R/R \to A_i \subseteq S^{-1}R/R$ denote the projection onto A_i. By part (a), for any $x \in S^{-1}R/R$, $\pi_i(x) \in xR \cap A_i$. This shows that $xR = \bigoplus_{i\in I}(xR \cap A_i)$ for any $x \in S^{-1}R/R$. Then the same is true for any submodule B of $S^{-1}R/R$.

(c) For the first statement, apply (b) to see that $B' = (A \cap B') \oplus (B \cap B') = B \cap B' \subseteq B$. By symmetry $B = B'$. For the second statement, let $S^{-1}R/R = A \oplus B = A' \oplus B'$. Then (b) yields $A + A' = (A \cap A') \oplus (A \cap B') \oplus (A' \cap B)$. So $(A + A') \cap (B \cap B') = 0$, and $(A + A') \oplus (B \cap B') = S^{-1}R/R$.

(d) For any $i \in I$, let $\pi_i\colon S^{-1}R/R \to A_i$ denote the canonical projection. Let $i, j \in I$, $i \neq j$, and let $f \in \mathrm{Hom}_R(A_i, A_j)$. We can apply part (a) to $f \circ \pi_i \in \mathrm{End}_R(S^{-1}R/R)$ to deduce that $f(x) \in xR \cap A_j \subseteq A_i \cap A_j = 0$ for any $x \in A_i$. Hence $f = 0$. ∎

We arrive at the notion of a restriction due to Hamsher [48]:

Definition 4.2. Let R be a commutative ring and M a module. A submodule $N \subseteq M$ is a *restriction* of M, if for each prime (equivalently, maximal) ideal p of R, the localization $N_{(p)}$ of N at p satisfies $N_{(p)} = 0$ or $N_{(p)} = M_{(p)}$.

An example of restriction is a direct summand of a cyclic module, because a cyclic module over a local ring is indecomposable. Proposition 4.4 gives another example of restriction. Both of them will be needed in Theorem 4.7.

Lemma 4.3. *Let R be a commutative ring. Let $N \subseteq M$ be R–modules such that N is a restriction of M. Let $s \in \Sigma$. If the multiplication by s is an onto endomorphism of M, then it is also an onto endomorphism of N. Equivalently, if M is $\{s\}$–divisible, then so is N.*

Proof. By the definition of a restriction, for any $p \in \operatorname{spec} R$, multiplication by s is an onto endomorphism of $N_{(p)}$. Thus it is an onto endomorphism of N. ■

Proposition 4.4. *Let R be a commutative ring and S a multiplicative subset of Σ.*

(a) *If R is local, then $S^{-1}R/R$ is indecomposable.*

(b) *Let A be an R–submodule of $S^{-1}R$ such that $R \subseteq A$. Assume that A/R is a direct summand in $S^{-1}R/R$. Then A/R is a restriction of $S^{-1}R/R$.*

(c) *Let A be an R–submodule of $S^{-1}R$ such that $R \subseteq A$. Assume that A/R is a restriction of $S^{-1}R/R$. Then A is a subring of $S^{-1}R$.*

Proof. (a) Since R is local, all cyclic modules are indecomposable, hence $S^{-1}R/R$ is indecomposable (see e.g. [63, Theorem 4.7]).

(b) If A/R is a direct summand of $S^{-1}R/R$, then, for any $p \in \operatorname{spec} R$, $(A/R)_{(p)}$ is a direct summand of $(S^{-1}R/R)_{(p)}$. By part (a) either $(A/R)_{(p)} = 0$ or $(A/R)_{(p)} = (S^{-1}R/R)_{(p)}$. Then A/R is a restriction of $S^{-1}R/R$, and (b) is proved.

To prove (c), let $X = \{p \in \operatorname{spec} R \mid (A/R)_{(p)} \neq 0\}$ be the support of A/R. For each $p \in \operatorname{spec} R$, denote by $\alpha_p\colon S^{-1}R \to (S^{-1}R/R)_{(p)}$ the canonical ring homomorphism. Assume that $\frac{x}{s}$ is an element of $S^{-1}R$ such that $\alpha_p(\frac{x}{s}) = 0$ for all $p \in \operatorname{spec} R \setminus X$. Then, since $(S^{-1}R/R)_{(p)} = (A/R)_{(p)}$ for all $p \in X$, we deduce that $\alpha_p(\frac{x}{s}) \in (A/R)_{(p)}$ for all $p \in \operatorname{spec} R$. This implies that for each $p \in \operatorname{spec} R$ there is an element $t \in R \setminus p$ such that $t\,\frac{x}{s} \in A$ and therefore proves that $\frac{x}{s} \in A$. Conversely, it is clear that $\alpha_p(A) = (A/R)_{(p)} = 0$ for all $p \in \operatorname{spec} R \setminus X$. We thus conclude that $A = \bigcap_{p \notin X} \operatorname{Ker} \alpha_p$, so A is a ring. ■

We will next investigate direct sum decompositions of $S^{-1}R/R$ under the assumption of $\operatorname{proj\,dim} S^{-1}R \leq 1$.

Note that localizations of projective dimension at most one are rather frequent. For instance, if R is any commutative ring and $S = \{1 = s_0, s_1, \dots\}$ is a countable multiplicative subset of Σ, then $\operatorname{pd}(S^{-1}R) \leq 1$. This follows by Lemma 1.30 applied to the filtration $(\frac{1}{s_0 \dots s_i} R \mid i < \omega)$ of $S^{-1}R = \bigcup_{i<\omega} \frac{1}{s_0 \dots s_i} R$.

The next proposition goes back to Hamsher:

Proposition 4.5. *Let R be a commutative ring. Let S be a multiplicative subset of Σ such that $S^{-1}R$ has projective dimension at most 1. Let A be a non–zero R–submodule of $S^{-1}R$ such that* $\operatorname{proj\,dim} \big(S^{-1}R/A\big) \leq 1$. *Then A/sA is R/sR–projective for any $s \in S$. If, in addition, R is local and A is divisible by a non–unit in S, then $A = S^{-1}R$.*

Proof. Let $Q = S^{-1}R$ and $s \in S$. By assumption, $\operatorname{proj\,dim} Q \leq 1$ and $\operatorname{proj\,dim} Q/A \leq 1$, so $\operatorname{proj\,dim} A \leq 1$. Then it is well–known that the projective dimension of A/sA viewed as R/sR–module is also ≤ 1. Since $Q/sA \cong Q/A$, the exact sequence $0 \to A/sA \to Q/sA \to Q/A \to 0$ yields that $\operatorname{proj\,dim} A/sA \leq 1$. Now, if the projective dimension of A/sA viewed as R/sR–module equals 1, then $\operatorname{proj\,dim} A/sA = 2$, a contradiction. So A/sA is R/sR–projective.

For the second claim it suffices to show that $A = tA$ for each $t \in S$. By the first part, A/tA is a projective R/tR–module, so, since R is local, A/tA is a free R/tA–module divisible by a non–unit in R/tR, hence $A/tA = 0$. ■

As proved in Proposition 4.4 the direct sum decompositions of $S^{-1}R/R$ are parameterized by certain subsets of $\operatorname{spec} R$. In the next lemma, we describe the support of $S^{-1}R/R$. (Recall that the *support* of a module M is defined as the set of all $p \in \operatorname{mspec} R$ such that $M_{(p)} \neq 0$; the support is denoted by $\operatorname{supp} M$.)

Lemma 4.6. *Let R be a commutative ring. Let S be a multiplicative subset of Σ, and $p \in \operatorname{spec} R$. Then $(S^{-1}R/R)_{(p)} = 0$, if and only if $p \in V(S)$. In particular, we have* $\operatorname{supp}\,(S^{-1}R/R) = \{p \in \operatorname{mspec} R \mid p \cap S \neq \emptyset\}$.

Proof. If $p \in V(S)$, then $S \subseteq R \setminus p$. Thus $(S^{-1}R/R)_{(p)} \cong R_{(p)}/R_{(p)} = 0$. Let p be a prime ideal of R such that $p \notin V(S)$. Let $s \in p \cap S$. Then $\frac{1}{s} + R_{(p)}$ is a non–zero element of $S^{-1}(R_{(p)})/R_{(p)} = (S^{-1}R/R)_{(p)}$. ∎

The following theorem characterizes direct summands in $S^{-1}R/R$:

Theorem 4.7. *Let R be a commutative ring and S be a multiplicative subset of Σ such that* $\operatorname{proj\,dim} S^{-1}R = 1$. *Let $M_1 = A_1/R$ be a submodule of $S^{-1}R/R$. Put $X_1 = $* supp $S^{-1}R/R$ and $X_2 = \operatorname{supp} M_1$. Let

$$\varphi\colon S^{-1}R/R \to \prod_{p \in \operatorname{spec} R \setminus V(S)} (S^{-1}R/R)_{(p)}$$

be the canonical inclusion. Put $M_2 = \varphi^{-1}(\prod_{p \in X_2} (S^{-1}R/R)_{(p)}) = A_2/R$. Then the following conditions are equivalent:

(a) M_1 is a direct summand of $S^{-1}R/R$.

(b) $\operatorname{proj\,dim}(S^{-1}R/A_1) \leq 1$ and M_1 is a restriction of $S^{-1}R/R$.

(c) $S^{-1}R/R = M_1 \oplus M_2$.

Proof. Since $\operatorname{proj\,dim}(S^{-1}R) = 1$, Proposition 4.4 (b) yields the implication (a)⇒(b), and clearly (c) implies (a).

Assume (b). By the definition of M_1 and M_2, it follows that $M_1 \cap M_2 = 0$ and also that, for $i = 1, 2$, $(M_i)_{(p)} = 0$ for each maximal ideal $p \notin X_i$. We shall show that $S^{-1}R = A_1 + A_2$. Take $s \in S$. It is enough to prove that $\frac{1}{s} \in A_1 + A_2$.

By Lemma 4.3, s induces an onto map on $M_1 = A_1/R$, hence $sA_1 + R = A_1$. This implies that A_1/sA_1 is cyclic and that the map $\pi_1\colon R/sR \to A_1/sA_1$ defined by $r + sR \mapsto r + sA_1$, for $r \in R$, is surjective. By Proposition 4.5, A_1/sA_1 is a projective R/sR–module, so π_1 splits. Thus there exists $a \in R$ such that $a - a^2 \in sR$, $A_1 = aR + sA_1$ and $1 - a \in sA_1$. So $\frac{1-a}{s} \in A_1$. We show that $\frac{a}{s} \in A_2$. This occurs if and only if $\varphi(\frac{a}{s} + R) \in \prod_{p \in X_2} (S^{-1}R/R)_{(p)}$, if and only if $((a + sR)R/sR)_{(p)} = 0$ for each $p \in X_1 \cup (\operatorname{spec} R \setminus X)$.

As $R/sR = (a + sR)R/sR \oplus (1 - a + sR)R/sR$, $(a + sR)R/sR$ and $(1 - a + sR)R/sR$ are restrictions of R/sR. For any $p \in \operatorname{spec} R \setminus X$, as $s \in S$, $(R/sR)_{(p)} = 0$, hence also $((a + sR)R/sR)_{(p)} = 0$. Let $p \in X_1$. As $(A_1)_{(p)} = (S^{-1}R)_{(p)}$ and $s \in S$, $0 = (A_1/sA_1)_{(p)} \cong ((a + sR)R/sR)_{(p)}$. This finishes the proof of (c). ∎

Proposition 4.8. *Let R be a commutative ring. Let $S_1 \subseteq S$ be multiplicative subsets of Σ such that* $\operatorname{proj\,dim}(S^{-1}R) = 1$ *and* $\operatorname{proj\,dim}(S^{-1}R/S_1^{-1}R) \leq 1$. *Then $S_1^{-1}R/R$ is a restriction of $S^{-1}R/R$. More precisely,*

(a) *If $p \in V(S_1)$, then $(S_1^{-1}R/R)_{(p)} = 0$.*

(b) *If $p \in \operatorname{spec} R \setminus V(S_1)$, then $(S_1^{-1}R/R)_{(p)} = (S^{-1}R/R)_{(p)}$.*

Proof. (a) follows from Lemma 4.6. If $p \in \operatorname{spec} R \setminus V(S_1)$, then there exists a non–unit $s \in S_1 \cap p \subseteq S$. Then Proposition 4.5 yields

$$(S_1^{-1}R/R)_{(p)} = S_1^{-1}(R_{(p)})/R_{(p)} = S^{-1}(R_{(p)})/R_{(p)} = (S^{-1}R/R)_{(p)}.$$

■

In view of Theorem 4.7 we immediately obtain

Corollary 4.9. *Let R be a commutative ring and $S_1 \subseteq S$ a multiplicative subset of Σ such that* $\operatorname{proj\,dim} S^{-1}R = 1$. *Then* $\operatorname{proj\,dim}(S^{-1}R/S_1^{-1}R) \leq 1$, *if and only if $S_1^{-1}R/R$ is a direct summand in $S^{-1}R/R$.*

Example 4.10. By Proposition 4.4, the complement of the module $S_1^{-1}R/R$ in the statement of Corollary 4.9 is of the form A/R for a subring A of $S^{-1}R$ containing R.

However, A is not a localization in general: consider the case when R is a Dedekind domain, $S = R \setminus \{0\}$, and $R \subseteq A \subseteq Q = RS^{-1}$ is defined by $A/R \cong E(R/p)$, where p is a maximal ideal contained in the union of all the maximal ideals $q \neq p$ (such a p exists iff the class group of R contains torsion–free elements).

Then $A \oplus R_{(p)}/R = Q/R$, but A is not of the form $(S')^{-1}R$ for a multiplicative subset $S' \subseteq S$: otherwise, since $A = \bigcap_{q \neq p} R_{(q)}$, we have $S' \subseteq \bigcap_{q \neq p}(R \setminus q) = R \setminus (\bigcup_{q \neq p} q) = R^*$, where R^* is the set of all units of R, so $(S')^{-1}R = R \neq A$, a contradiction.

We will also need Griffith's notion of a $G(\aleph_0)$–family of submodules:

Definition 4.11. Let R be a commutative ring and M a module. A family $\mathcal{S}$ of submodules of M is a $G(\aleph_0)$*–family* provided that

(G1) $0, M \in \mathcal{S}$,

(G2) $\mathcal{S}$ is closed under unions of chains, and

(G3) if $N \in \mathcal{S}$ and X is a countable subset of M, then there exists $N' \in \mathcal{S}$ such that $N \cup X \subseteq N'$ and N'/N is countably generated.

Lemma 4.12. *Let R be a commutative ring, S a multiplicative subset of Σ and $Q = S^{-1}R$. Assume* $\operatorname{proj\,dim} Q \leq 1$. *Then the set $\mathcal{S}$ of all restrictions of Q/R is a $G(\aleph_0)$–family of submodules of Q/R.*

Proof. Property (G1) is clear, and (G2) follows from Definition 4.2, because $N_{(p)} = N \otimes_R R_{(p)}$, where $R_{(p)}$ is a flat R–module.

Property (G3) is a consequence of the following claim: for each $N \in \mathcal{S}$ and each countable subset $X \subseteq Q/R$, there is a countable multiplicative subset $S_1 \subseteq S$ such that $X \subseteq S_1^{-1}R/R$ and $\operatorname{proj\,dim} Q/S_1^{-1}R \leq 1$. Indeed, Proposition 4.8 then shows that $S_1^{-1}R/R$ is a restriction of Q/R, and so is $N' = N + S_1^{-1}R/R$. Since N'/N is countably generated, (G3) follows.

In order to prove the claim, we consider a projective resolution of Q

$$0 \to K \xrightarrow{\subseteq} F \xrightarrow{f} Q \to 0,$$

where $F = R^{(S)}$, $f(1_s) = s^{-1}$ for each $s \in S$ (where $(1_s \mid s \in S)$ is the canonical basis of F).

Notice that, if T is a countable multiplicative subset of S, then $K \cap R^{(T)}$ is generated by the countable set $K_T = \{1_t - t'1_{t.t'} \mid t, t' \in T\}$: indeed, these elements clearly belong to K. If $x \in K \cap R^{(T)}$, $x = 1_{t_0}r_0 + \cdots + 1_{t_k}r_k$, then we can w.l.o.g. assume that $t_0 = 1$ and, for each $m < k$, $t_{m+1} = t_m.u_{m+1}$ for some $u_{m+1} \in T$. Since $f(x) = t_0^{-1}r_0 + \cdots + t_k^{-1}r_k = 0$, we have $(u_1 \dots u_k)r_0 + \cdots + u_k r_{k-1} + r_k = 0$. So

$$x = 1_{t_0}r_0 + \cdots + 1_{t_{k-1}}r_{k-1} + 1_{t_k}r_k - 1_{t_k}(u_1 \dots u_k r_0) - \cdots - 1_{t_k}u_k r_{k-1} - 1_{t_k}r_k$$

$$= (1_{t_0} - 1_{t_k}(u_1 \dots u_k))r_0 + \cdots + (1_{t_{k-1}} - 1_{t_k}u_k)r_{k-1},$$

which shows that x is generated by the elements of K_T.

Since K is projective, $K = \bigoplus_{j \in J} K_j$ where K_i is countably generated (see Corollary 2.24).

By induction, we define an increasing sequence $(T_i \mid i < \omega)$ of countable multiplicative subsets of S as follows: T_0 is any countable multiplicative subset of S such that $X \subseteq RT_0^{-1}/R$. If T_i is defined, then $K \cap T_i \subseteq \bigoplus_{j \in A_i} K_j$ for a countable set $A_i \subseteq J$. Let B_i be a countable subset of S such that $\bigoplus_{j \in A_i} K_j \subseteq R^{(B_i)}$. Define T_{i+1} as a countable multiplicative subset of S containing $T_i \cup B_i$. Finally, put $S_1 = \bigcup_{i<\omega} T_i$. Then S_1 is a countable multiplicative subset of S such that $X \subseteq RS_1^{-1}/R$. Moreover, we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \cap R^{(S_1)} \arrow[r, "\subseteq"] \arrow[d, "\subseteq"'] & K \arrow[r] \arrow[d, "\subseteq"'] & K/(K \cap R^{(S_1)}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R^{(S_1)} \arrow[r, "\subseteq"] \arrow[d] & F \arrow[r] \arrow[d] & R^{(S \setminus S_1)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & RS_1^{-1} \arrow[r, "\subseteq"] & Q \arrow[r] \arrow[d] & Q/RS_1^{-1} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

By construction, $K \cap R^{(S_1)} = \bigoplus_{j \in \bigcup_{i<\omega} A_i} K_j$ is a direct summand in K, so the right hand column gives $\operatorname{proj\,dim} Q/S_1^{-1}R \leq 1$. ∎

Recall that given a commutative ring R and a set $S \subseteq R$ consisting of non–zero divisors, then the class of all S*–divisible modules*, $\mathcal{D}_S = \{M \in \text{Mod–}R \mid$

$Ms = M$ for all $s \in S\}$, satisfies $\mathcal{D}_S = \mathcal{S}^{\perp}$, where $\mathcal{S} = \{R/sR \mid s \in S\} \subseteq \mathcal{P}_1^{<\omega}$. By Theorem 3.15, $\mathcal{D}_S$ is a 1–tilting torsion class.

Now we are in a position to prove the main result of this chapter:

Theorem 4.13. *Let R be a commutative ring and S a multiplicative subset in R consisting of non–zero–divisors. Then the following conditions are equivalent:*

- (a) *$S^{-1}R$ is a Matlis localization of R.*
- (b) *$T = S^{-1}R \oplus S^{-1}R/R$ is a 1–tilting R–module.*
- (c) *$S^{-1}R/R$ decomposes into a direct sum of countably presented R–submodules.*
- (d) *R has a $\mathcal{D}_S$–envelope.*

Moreover, if T is 1–tilting then T generates the 1–tilting class $\mathcal{D}_S$.

Proof. Let $Q = S^{-1}R$. First we prove the equivalence of (a)–(c):

Assume (a). We will verify conditions (T1)–(T3) for T, thus proving (b). First the projective dimension of Q, Q/R, and hence of T, is ≤ 1 by the assumption, so (T1) holds. (T3) holds, since there is the exact sequence $0 \to R \to Q \to Q/R \to 0$. In order to prove (T2), in view of (T1), it suffices to show that $\mathrm{Ext}^1_R(Q/R, Q^{(\kappa)}) = 0$ for each cardinal κ. However, $\mathrm{Ext}^1_R(Q, Q^{(\kappa)}) \cong \mathrm{Ext}^1_Q(Q, Q^{(\kappa)}) = 0$, since Q is a localization of R. So in order to prove that $\mathrm{Ext}^1_R(Q/R, Q^{(\kappa)}) = 0$, it suffices to show that any $f \in \mathrm{Hom}_R(R, Q^{(\kappa)})$ extends to some $g \in \mathrm{Hom}_R(Q, Q^{(\kappa)}) = \mathrm{Hom}_Q(Q, Q^{(\kappa)})$. But we can simply define $g(q) = f(1)q$ for all $q \in Q$.

Next, assume (b). Consider the cotorsion pair $(\mathcal{A}, \mathcal{B})$ generated by T. By Theorem 3.16, each module in $\mathcal{A}$ is $\mathcal{A}^{\leq\omega}$–filtered. In particular, this holds for $Q/R \in \mathcal{A}$. Let $\mathcal{F}$ be a family corresponding to a $\mathcal{A}^{\leq\omega}$–filtration of Q/R by Theorem 2.20 (for $\kappa = \aleph_1$). Let $\mathcal{G} = \mathcal{F} \cap \mathcal{S}$ where $\mathcal{S}$ is the $G(\aleph_0)$–family of all restrictions of Q/R coming from Lemma 4.12.

We claim that there is a filtration $(G_\alpha \mid \alpha \leq \sigma)$ of Q/R such that $G_\alpha \in \mathcal{G}$ for all $\alpha \leq \sigma$ and $G_{\alpha+1}/G_\alpha$ is countably presented. Indeed, we can define $G_0 = 0$ and $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ for limit ordinals α. Assume $G_\alpha \in \mathcal{G}$ is defined and there is $x \in (Q/R) \setminus G_\alpha$. Let $F_0 = S_0 = G_\alpha$. By Theorem 2.20, there is $F_1 \in \mathcal{F}$ such that $F_0 \cup \{x\} \subseteq F_1$ and F_1/F_0 is countably presented. Clearly $S_0 \subseteq F_1$. Let C_1 be a countable subset of F_1 such that $F_0 + \langle C_1 \rangle = F_1$. Since $\mathcal{S}$ is a $G(\aleph_0)$–family, there $S_1 \in \mathcal{S}$ such that $S_0 \cup C_1 \subseteq S_1$ and S_1/S_0 is countably generated. Then $F_1 \subseteq S_1$. Let D_1 be a countable subset of S_1 such that $S_0 + \langle D_1 \rangle = S_1$. Then there is $F_2 \in \mathcal{F}$ such that $F_1 \cup D_1 \subseteq F_2$ and F_2/F_1 is countably presented. Then $S_1 \subseteq F_2$. Proceeding in this way, we obtain a chain

$$G_\alpha = F_0 = S_0 \subseteq F_1 \subseteq S_1 \subseteq F_2 \subseteq \cdots \subseteq S_n \subseteq F_{n+1} \subseteq S_{n+1} \subseteq \ldots$$

We define $G_{\alpha+1} = \bigcup_{n<\omega} F_n = \bigcup_{n<\omega} S_n$. Then $G_{\alpha+1} \in \mathcal{G}$, and since F_{n+1}/F_n is countably presented for each $n < \omega$, so is $G_{\alpha+1}/G_\alpha$ by Lemma 2.29. This proves the claim.

Now each G_α is a restriction of $Q/R = G_\sigma$ such that $(Q/R)/G_\alpha \in \mathcal{A}$, so $(Q/R)/G_\alpha$ has projective dimension ≤ 1. By Theorem 4.7, G_α is a direct summand in Q/R, and hence in $G_{\alpha+1}$, for each $\alpha < \sigma$. This yields a decomposition of Q/R into a direct sum of copies of the countably presented modules $G_{\alpha+1}/G_\alpha$ ($\alpha < \sigma$), so (c) holds.

Assume (c), so $Q/R = \bigoplus_{i\in I} M_i$ where M_i is a countably presented R–module for each $i \in I$.

First we claim that for each $i \in I$, M_i is a direct summand of a module of the form ${T_i}^{-1}R/R$, where T_i is a countable multiplicative subset of S.

To prove the claim set $J_0 = \{i\}$. By induction, we can construct an ascending chain $\{J_n\}_{n\geq 0}$ of countable subsets of I and an ascending chain $\{S_n\}_{n\geq 0}$ of countable multiplicative subsets of S such that $\bigoplus_{j\in J_n} M_j \subseteq S_n^{-1}R/R \subseteq \bigoplus_{j\in J_{n+1}} M_j$ for all $n \geq 0$. Let $J = \bigcup_{n\geq 0} J_n$ and $T_i = \bigcup_{n\geq 0} S_n$. Then $\bigoplus_{j\in J} M_j = T_i^{-1}R/R$, and T_i is a countable multiplicative subset of S. This proves the claim.

Now, since T_i is countable, ${T_i}^{-1}R$ is a countably generated flat R–module, hence ${T_i}^{-1}R$ is countably presented and has projective dimension ≤ 1 (see [44, VI.9]). Then we also have $\operatorname{proj\,dim} {T_i}^{-1}R/R \leq 1$, hence $\operatorname{proj\,dim} M_i \leq 1$ for each $i \in I$, and $\operatorname{proj\,dim} S^{-1}R/R \leq 1$, that is, $\operatorname{proj\,dim} S^{-1}R \leq 1$ and (a) holds.

Notice that $T = Q \oplus Q/R$ satisfies $\operatorname{Gen}(T) = \operatorname{Gen}(Q) \subseteq \mathcal{D}_S$, since $Q \in \mathcal{D}_S$ and $\mathcal{D}_S$ is a torsion class. If moreover T is 1–tilting, then $\operatorname{Gen}(T) = T^\perp$ (see Lemma 3.28), so by the reasoning above, $\operatorname{Gen}(T) = (Q/R)^\perp = \bigcap_{i\in I} M_i^\perp \supseteq \bigcap_{i\in I}(T_i^{-1}R/R)^\perp$, where T_i ($i \in I$) are countable multiplicative subsets of S. However, if $T = \{t_k \mid k < \omega\}$ is a countable multiplicative subset of S, then $T^{-1}R = \bigcup_{k<\omega}(t_0^{-1}\ldots t_k^{-1})R$ and $(t_0^{-1}\ldots t_{k+1}^{-1})R/(t_0^{-1}\ldots t_k^{-1})R \cong R/t_{k+1}R$ for each $k < \omega$, and $t_0^{-1}R/R \cong R/t_0R$, so $(T^{-1}R/R)^\perp \supseteq \mathcal{D}_S$ by Lemma 1.30. This proves that $(Q/R)^\perp \supseteq \mathcal{D}_S$, and hence $\operatorname{Gen}(T) = \mathcal{D}_S$.

Finally, we prove that (d) is equivalent to (a)–(c). Indeed, if (a)–(c) hold, then the embedding $\mu : R \to Q$ is a special $\mathcal{D}_S$–preenvelope of R. Since Q is a localization of R, we have $\operatorname{Hom}_R(Q,Q) = \operatorname{Hom}_Q(Q,Q)$, so the only R–endomorphism of Q fixing 1 is the identity. Thus, μ is left minimal, and μ is a $\mathcal{D}_S$–envelope of R.

Conversely, assume (d) and let $f\colon R \to D$ be the S–divisible envelope of R. First we show that D is S–torsion–free. To this end, we take $s \in S$ and show that the multiplication $\psi\colon D \to D$ by the element s is injective. We know that there is $d \in D$ such that $sd = f(1)$. Define an R–homomorphism $g\colon R \to D$ by $g(1) = d$. Since f is a $\mathcal{D}_S$–preenvelope, there is a map $\phi\colon D \to D$ such that $\phi f = g$. So $\phi\psi f(1) = s(\phi f(1)) = sg(1) = sd = f(1)$. By the left minimality of f we conclude that $\phi\psi$ is an isomorphism, hence ψ is injective.

It follows that D is an S–torsion–free, S–divisible module, hence a Q–module. In particular $D \in \operatorname{Gen}(Q)$. Moreover, since f is an S–divisible preenvelope, each epimorphism $R^{(\lambda)} \to X$ with X S–divisible factors through $f^{(\lambda)} : R^{(\lambda)} \to D^{(\lambda)}$. So D generates $\mathcal{D}_S$. This shows that $\mathcal{D}_S$ is contained in, and therefore coincides with, $\operatorname{Gen}(Q)$.

Let T' be a 1–tilting module generating the class $\mathcal{D}_S$. Since $\operatorname{Gen}(Q) = \operatorname{Gen}(T')$, there exist cardinals κ and λ and R–epimorphisms $f : (T')^{(\kappa)} \to Q$ and $g : Q^{(\lambda)} \to (T')^{(\kappa)}$. Then $fg : Q^{(\lambda)} \to Q$ is an R–epimorphism and hence a Q–epimorphism. So fg, and also f, splits, and the R–module Q is isomorphic to a direct summand in $(T')^{(\kappa)}$. Then Q has projective dimension ≤ 1, and (a) holds. ■

Particular instances of Theorem 4.13 include the classical result of Kaplansky characterizing Matlis valuation domains as the valuation domains with Q countable (this follows by taking R a valuation domain and $S = R \setminus \{0\}$), the theorem of Lee characterizing Matlis domains by the existence of a direct decomposition of Q/R into countably generated summands (take R a domain and $S = R \setminus \{0\}$), and the theorem of Fuchs and Salce (= the particular case when R is a domain and S a multiplicative subset of R).

By Theorem 3.29, the class of all S–divisible modules is a 1–tilting class for any commutative ring R. It is possible to generalize the notion of a Fuchs tilting module defined for domains in Example 3.2 to the setting of general commutative rings, so that the resulting 1–tilting module δ'_S generates the class of all S–divisible modules (see [5, §5]). Of course, in the case of Matlis localizations, δ'_S is equivalent to the module $S^{-1}R \oplus S^{-1}R/R$ by Theorem 4.13.

Theorem 4.13 is helpful in answering the question of the existence of minimal versions of preenvelopes in particular cases. We start with its immediate corollary:

Corollary 4.14. *Let R be a domain. Then R has a $\mathcal{DI}$-envelope, if and only if R is a Matlis domain.*

Notice that since $\mathcal{DI}$ is a tilting torsion class over any domain, tilting approximations need not have minimal versions (this contrasts with the dual setting of cotilting approximations, where the minimal versions always exist, see Theorem 3.41).

By Theorem 2.12, $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is a complete cotorsion pair. It appears open to determine when the $\mathcal{P}_n^{\perp}$–preenvelopes have minimal versions (= envelopes) for $n \geq 1$.

Our next corollary will give an answer for $n = 1$ in the particular case of Prüfer domains that are not Matlis. In this case, also fp–injective envelopes do not exist in general:

Corollary 4.15. *Assume R is a Prüfer domain.*

(a) *The cotorsion pair generated by δ is $(\mathcal{P}_1, \mathcal{DI})$ and $\mathcal{DI} = \mathcal{FI}$.*

(b) *If* $\operatorname{proj\,dim} Q \geq 2$*, then R does not have an $\mathcal{FI}$–envelope.*

Proof. (a) The first equality has already been considered in Example 3.2.

Since any Prüfer domain is coherent, each finitely generated submodule of a finitely presented module is finitely presented. So, if F is finitely presented, then there exist $n < \omega$ and a chain of submodules $0 = F_0 \subset F_1 \subset \cdots \subset F_n = F$ such that F_{i+1}/F_i is cyclic and finitely presented for all $i < n$. So

$$\mathcal{FI} = \{\bigoplus_{I \in \mathcal{F}} R/I\}^{\perp} = \mathcal{DI}$$

where $\mathcal{F}$ denotes the set of all finitely generated ideals of R.

(b) By part (a) and Corollary 4.14. ∎

5 Application 2: Finitistic Dimension Conjectures

In this chapter we present applications of tilting approximations to computing finitistic dimensions of rings and algebras.

The simple, but key fact is that the little finitistic dimension of a right noetherian ring is finite, if and only if there is a (possibly infinitely generated) tilting module T_f such that $T_f^{\perp_\infty} = (\mathcal{P}^{<\omega})^\perp$ (see Theorem 5.9 below). The surprising phenomenon here is that even in the artin algebra case, we cannot in general take T_f finitely generated, so the infinite–dimensional tilting theory developed above comes up as a natural tool.

Our first application concerns (non–commutative) Iwanaga–Gorenstein rings. In Theorem 5.11 we prove that, if R is n–Iwanaga–Gorenstein, then $\operatorname{fin\,dim} R = \operatorname{Fin\,dim} R = n$.

In the second application (Theorem 5.13), for a right artinian ring R, we provide a formula for computing $\operatorname{fin\,dim} R$ involving only approximations of the (finitely many) simple modules.

Our third application yields a simple proof of a result by Auslander, Reiten, Huisgen–Zimmermann and Smalø saying that $\operatorname{fin\,dim} R = \operatorname{Fin\,dim} R < \infty$ in case R is an artin algebra such that $\mathcal{P}^{<\omega}$ is contravariantly finite.

This chapter is based on [4] and [7].

Recall that the (right) *global dimension* of R, gl dim$R$, is the supremum of the projective dimensions of all (right R–) modules.

Definition 5.1. Let R be a ring. Denote by $\operatorname{Fin\,dim} R$ the *big finitistic dimension* of R, that is, the supremum of the projective dimensions of arbitrary modules of finite projective dimension.
Similarly, $\operatorname{fin\,dim} R$ will denote the *little finitistic dimension* of R, that is, the supremum of the projective dimensions of all finitely generated modules of finite projective dimension.
Obviously, $\operatorname{fin\,dim} R \leq \operatorname{Fin\,dim} R \leq \operatorname{gl\,dim} R$ for any ring R.

We recall a couple of simple and well–known facts:

Lemma 5.2. *Let R be a ring such that* $\operatorname{gl\,dim} R < \infty$.

(a) $\operatorname{fin\,dim} R = \operatorname{Fin\,dim} R = \operatorname{gl\,dim} R = \max\{\operatorname{proj\,dim} R/I \mid I \subseteq R\}$.

(b) *If R is right semiartinian, then all these dimensions are also equal to* $\max\{\operatorname{proj\,dim} S \mid S \in \operatorname{simp} R\}$.

Proof. This is an easy consequence of Lemma 1.30. ∎

So the little and the big finitistic dimensions provide a refinement of the homological dimension theory in the case when $\operatorname{gl\,dim} R = \infty$. The following example shows that such refinement is needed even in very simple cases:

Example 5.3. Let R be a quasi–Frobenius (= 0–Iwanaga–Gorenstein) ring which is not completely reducible. (For example, let p be a prime integer, $n > 1$ and $R = \mathbb{Z}_{p^n}$.) Since all projective modules are injective, there is no module of projective dimension 1, hence no module of projective dimension m for any $m \geq 1$. By assumption, there is a non–projective simple module M, so $\operatorname{proj\,dim} M = \infty$. It follows that $\operatorname{fin\,dim} R = \operatorname{Fin\,dim} R = 0$, while $\operatorname{gl\,dim} R = \infty$. (Since $R = \mathbb{Z}_{p^n}$ is of finite representation type, it is certainly the finitistic dimension rather than the global one that reflects better the transparent structure of the module category Mod–$\mathbb{Z}_{p^n}$.)

Notice that, if R is a right $\aleph_0$–noetherian ring, then the possible difference between $\operatorname{Fin\,dim} R$ and fin dim$R$ comes from (a representative set of) *countably* infinitely generated modules of finite projective dimension:

Lemma 5.4. *Assume that each right ideal of R is countably generated. Then* $\operatorname{Fin\,dim} R = \sup\{\operatorname{proj\,dim} M \mid M \in \mathcal{P}^{\leq\omega}\}$.

Proof. This follows by Theorem 2.12. ∎

Example 5.5. Let R be a commutative noetherian ring. Then the little and the big finitistic dimensions are known to be closely related to other dimensions of the ring. Bass, Gruson and Raynaud proved that $\operatorname{Fin\,dim} R$ coincides with the Krull dimension of R. Auslander and Buchsbaum proved that, if R is moreover local, then $\operatorname{fin\,dim} R = \operatorname{depth} R$, where $\operatorname{depth} R$ denotes the length of a maximal regular sequence in $\operatorname{Rad} R$. So in the local case, both dimensions are finite, but they coincide, if and only if R is Cohen–Macaulay. Examples of commutative noetherian rings with $\operatorname{Fin\,dim} R = \operatorname{fin\,dim} R = \infty$ were constructed by Nagata:

Let $R = K[x_i \mid i < \omega]$ be the polynomial ring in countably many variables over a field K. Let $(d_i \mid i < \omega) \subseteq \omega$ be a strictly increasing sequence of natural numbers such that $d_{i+1} - d_i > d_i - d_{i-1}$ for all $i > 0$. For each $i < \omega$, let $P_i = \langle x_{d_i+1}, \dots, x_{d_{i+1}} \rangle$ be the prime ideal in R generated by the variables x_j ($d_i < j \leq d_{i+1}$). Let $U = R \setminus \bigcup_{i<\omega} P_i$ and let S be the localization of R at U. Then S is noetherian, the Krull dimension of S is ∞, and $\operatorname{fin\,dim} S = \operatorname{Fin\,dim} S = \infty$. For more details we refer to [11, §11].

If R is an arbitrary ring, then the statements

(I) $\operatorname{Fin\,dim} R = \operatorname{fin\,dim} R$,

(II) $\operatorname{fin\,dim} R < \infty$

are known as the *first* and the *second finitistic dimension conjectures* for R, respectively.

In the case of artin algebras, (I) was disproved by Huisgen–Zimmermann: for each $n \geq 2$, she constructed a finite–dimensional monomial algebra Λ_n such that $\operatorname{fin\,dim} R = n$ and $\operatorname{Fin\,dim} R = n + 1$ (see [51]).

Examples with arbitrarily big differences between the two dimensions were later constructed by Smalø: for each $n \geq 1$ there is a finite–dimensional algebra R_n over a field such that $\operatorname{fin\,dim} R = 1$ and $\operatorname{Fin\,dim} R = n$.

The second finitistic dimension conjecture has been proved for all finite dimensional monomial algebras [47], all algebras with representation dimension ≤ 3 [55] et al., but it remains open for general artin algebras.

Now we fix our notation for the rest of this section:

Let R be a ring. Denote by $\mathfrak{C}_f = (\mathcal{A}_f, \mathcal{B}_f)$ the cotorsion pair generated by the class $\mathcal{P}^{<\omega}$. Recall that $\mathcal{P}^{<\omega} = \mathcal{P} \cap \text{mod–}R$, and $\mathcal{P}^{<\omega}$ is just the class of all finitely presented modules of finite projective dimension in case R is right coherent.

Since $\mathcal{P}^{<\omega}$ always has a representative set of elements, $\mathfrak{C}_f$ is complete. So $\mathcal{A}_f$ is a special precovering class, but — unlike $\mathcal{P}^{<\omega}$ — $\mathcal{A}_f$ always contains infinitely generated modules. However, $\mathcal{A}_f \subseteq \varinjlim \mathcal{P}^{<\omega}$ by Theorem 2.61, so $\mathcal{A}_f \cap \text{mod–}R = \mathcal{P}^{<\omega}$ by Lemma 2.55.

Tilting theory relates to the finitistic dimension conjectures by means of the following simple observation:

Lemma 5.6. *Let $n < \omega$. Let R be a right coherent ring and $\mathcal{S}$ be a syzygy closed class of finitely presented modules. Let $(\mathcal{U}, \mathcal{V})$ be the cotorsion pair generated by $\mathcal{S}$. Then the following assertions are equivalent:*

(a) $\mathcal{U} \subseteq \mathcal{P}_n$.

(b) *There exists a tilting module T of projective dimension $\leq n$ such that $\mathcal{V} = T^{\perp_\infty}$.*

Proof. Assume (a). By assumption, $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ and $\mathcal{S}^\perp = \mathcal{S}^{\perp_\infty}$, so $\mathcal{V}$ is a class of finite type, hence n–tilting by Theorem 3.15.
If (b) holds, then $\mathcal{U} = {}^\perp(T^{\perp_\infty}) \subseteq \mathcal{P}_n$. ∎

Varying the class $\mathcal{S}$, we get a rich supply of (infinitely generated) tilting modules:

Lemma 5.7. *Let R be a right coherent ring and $n < \omega$. Denote by $(\mathcal{A}_n, \mathcal{B}_n)$ the cotorsion pair generated by $\mathcal{P}_n^{<\omega}$. Then there is a tilting module T_n of projective dimension at most n such that $\mathcal{B}_n = T_n^{\perp_\infty}$.*
If R is right noetherian and $\operatorname{fin\,dim} R \geq n$, then T_n has projective dimension n.

Proof. The first assertion follows by Lemma 5.6 for $\mathcal{S} = \mathcal{P}_n^{<\omega}$.

Assume that R is right noetherian with $\operatorname{fin\,dim} R \geq n$. So there is a finitely presented module M of projective dimension n. Assume that there is a tilting module $T \in \mathcal{P}_{n-1}$ with $\mathcal{B}_n = T^{\perp_\infty}$. On the one hand, by Lemma 5.6 we then have $\mathcal{A}_n \subseteq \mathcal{P}_{n-1}$. On the other hand, $\mathcal{P}_n^{<\omega} \subseteq {}^\perp((\mathcal{P}_n^{<\omega})^\perp) = \mathcal{A}_n$. This implies that $M \in \mathcal{P}_{n-1}$, a contradiction. ∎

Example 5.8. Let R be a right coherent ring. For all $0 < i \leq n < \omega$, denote by $(\mathcal{A}_{ni}, \mathcal{B}_{ni})$ the cotorsion pair generated by the class $\Omega^{i-1}(\mathcal{P}_n^{<\omega})$. By Theorem 1.40 and Lemma 1.33, this cotorsion pair is complete and $\mathcal{A}_{ni} \subseteq \mathcal{P}_{n-i+1}$.

By dimension shifting, we have $\mathcal{B}_{ni} = (\mathcal{P}_n^{<\omega})^{\perp_i}$ for all $0 < i \leq n < \omega$. Note that $\mathcal{B}_{ni} \subseteq \mathcal{B}_{nj}$ for $i \leq j$, $\mathcal{B}_{ni} \subseteq \mathcal{B}_{mi}$ for $m \leq n$, and $\mathcal{B}_{ni} \subseteq \mathcal{B}_{n+k,i+k}$ for each $k < \omega$.
Clearly $\mathcal{A}_n = \mathcal{A}_{n1}$, $\mathcal{B}_n = \mathcal{B}_{n1}$ for all $1 \leq n < \omega$, and $\mathcal{B}_f = \bigcap_{n<\omega} \mathcal{B}_n$.

Let $0 < i \leq n < \omega$. Then Lemma 5.6 for $\mathcal{S} = \Omega^{i-1}(\mathcal{P}_n^{<\omega})$ yields a tilting module T_{ni} of projective dimension at most $n - i + 1$ such that $\mathcal{B}_{ni} = T_{ni}^{\perp_\infty}$.

In particular, the classes $\mathcal{B}_{nn}$ $(n < \omega)$ form an increasing chain of 1–tilting torsion classes. If R is right noetherian and $\operatorname{fin\,dim} R \geq n$, then, as in Lemma 5.7, we see that the projective dimension of T_{ni} equals $n - i + 1$.

It is easy to see that there is a single tilting module that controls the global dimension of the category of all finitely generated modules in the case when R is right noetherian and the latter dimension is finite:

Theorem 5.9. *Assume R is a right noetherian ring. Then* $\operatorname{fin\,dim} R < \infty$*, if and only if there is a tilting module T_f such that $\mathcal{B}_f = T_f^{\perp_\infty}$. In this case* $\operatorname{fin\,dim} R = \operatorname{proj\,dim} T_f$*, and T_f can be taken $\mathcal{P}^{<\omega}$–filtered.*

Proof. Assume that $\operatorname{fin\,dim} R = n < \infty$. Then $\mathcal{B}_f = \mathcal{B}_n$. By Lemma 5.7 there is a tilting module T_f of projective dimension n such that $\mathcal{B}_f = T_f^{\perp_\infty}$.
The reverse implication follows by Lemma 5.6 for $\mathcal{S} = \mathcal{P}^{<\omega}$.
Finally, T_f can be taken $\mathcal{P}^{<\omega}$–filtered by Theorem 3.25 (b). ■

Clearly the module T_f is unique up to the equivalence of tilting modules. In Theorem 5.19, we will see that even for finite–dimensional algebras with little finitistic dimension $= 1$, it need not be possible to select T_f finitely generated. The proof of Theorem 3.12 provides an explicit construction of T_f:
By a finite iteration of special $(\mathcal{P}^{<\omega})^{\perp}$–preenvelopes (of R etc.), we obtain a $(\mathcal{P}^{<\omega})^{\perp}$–coresolution of R. The tilting module T_f can simply be taken as the direct sum of the members of this finite coresolution.

The problem is that, in general, it is rather difficult to compute these special $(\mathcal{P}^{<\omega})^{\perp}$–preenvelopes, and hence determine T in this way. However, there are a number of cases when T can be computed explicitly. We start with a very simple one:

Example 5.10. Let R be a right noetherian ring of finite global dimension. Let $0 \to R \to I_0 \to I_1 \to \ldots \to I_n \to 0$ be the minimal injective coresolution of R. Then $T_f = \bigoplus_{i\leq n} I_i$ is a tilting module with $\operatorname{proj\,dim} T_f = \operatorname{gl\,dim} R = \operatorname{fin\,dim} R$ such that $T_f^{\perp_\infty} = (\mathcal{P}^{<\omega})^{\perp}$. Indeed, in this case $\mathcal{P}^{<\omega} = \text{mod–}R$, so $(\mathcal{P}^{<\omega})^{\perp}$–envelopes coincide with the injective envelopes.

There is another, non–trivial case where the tilting module T_f can be taken as the direct sum of the terms of the minimal injective coresolution of R. This is the case of (non–commutative) Iwanaga–Gorenstein rings considered in Examples 2.14 and 3.6. The explicit knowledge of T_f here yields a proof of the first finitistic dimension conjecture:

Theorem 5.11. *Let $n \geq 0$ and R be an n–Iwanaga–Gorenstein ring. Then*

(a) $\operatorname{fin\,dim} R = \operatorname{Fin\,dim} R = n$*.*

(b) $T_f = \bigoplus_{i\leq n} I_i$ *where* $0 \to R \to I_0 \to I_1 \to \ldots \to I_n \to 0$ *is the minimal injective coresolution of R.*

(c) $\mathcal{A}_f = \mathcal{P} = \mathcal{P}_n = \mathcal{I}_n = \mathcal{I}$, $\mathcal{B}_f = \mathcal{GI}$ *is the class of all $\mathcal{I}_0$–resolved modules, and* $\operatorname{Add}(T_f) = \mathcal{I}_0$*.*

Proof. First note that by Example 2.14, $\mathcal{P} = \mathcal{P}_n = \mathcal{I}_n = \mathcal{I}$, so $\operatorname{Fin\,dim} R = n$.
By Example 3.6, $T = \bigoplus_{i\leq n} I_i$ is an n–tilting module. Denote by $(\mathcal{A}, \mathcal{B})$ the tilting cotorsion pair induced by T. Since T is injective and R is right noetherian, clearly $\operatorname{Add}(T) \subseteq \mathcal{I}_0$.

We will prove that $\operatorname{Add}(T) = \mathcal{I}_0$. Since $\operatorname{Add}(T) = \mathcal{A} \cap \mathcal{B}$ by Lemma 3.8 (c), it suffices to prove that $\mathcal{I}_0 \subseteq \mathcal{A}$. By Proposition 3.9 (b), each module $B \in \mathcal{B}$ is $\operatorname{Add}(T)$–resolved. Denote by B' the kernel of the n–th map in a fixed $\operatorname{Add}(T)$–resolution of B. Since $\operatorname{Add}(T) \subseteq \mathcal{I}_0$, dimension shifting gives

$\mathrm{Ext}^1_R(I, B) \cong \mathrm{Ext}^{n+1}_R(I, B')$ for each $I \in \mathcal{I}_0$. However, $\mathcal{I}_0 \subseteq \mathcal{P}_n$, so the latter Ext–group is zero, proving that $I \in \mathcal{A}$.

Since $\mathrm{Add}(T) = \mathcal{I}_0$, Proposition 3.9 yields that $\mathcal{A} = \mathcal{I}_n\,(= \mathcal{P})$, and $\mathcal{B}$ is the class of all $\mathcal{I}_0$–resolved modules. However, by Example 2.14, $\mathcal{B} = \mathcal{P}^\perp = \mathcal{GI}$ is the class of all Gorenstein injective modules.

Finally, since T is a tilting module, T is of finite type, so $\mathcal{B} = (\mathcal{A}^{<\omega})^\perp$ (see Theorem 3.25). However, $\mathcal{A}^{<\omega} = \mathcal{P}^{<\omega}$, so $(\mathcal{A}, \mathcal{B}) = (\mathcal{A}_f, \mathcal{B}_f)$ and $T = T_f$. By Corollary 1.42, any module $P \in \mathcal{P}$ is a direct summand of a $\mathcal{P}^{<\omega}$–filtered module, so we conclude that $\mathrm{fin\,dim}\, R = \mathrm{Fin\,dim}\, R$. ∎

Let R be a right noetherian ring with $\mathrm{fin\,dim}\, R < \infty$. Then clearly, a sufficient condition for $\mathrm{fin\,dim}\, R = \mathrm{Fin\,dim}\, R$ to hold is

(III) $\mathcal{A}_f = \mathcal{P}$.

All our proofs of the first finitistic dimension conjecture for a ring R proceed by proving (III) (see Theorems 5.11 and 5.21). However, (III) is not necessary for the first finitistic dimension conjecture to hold, even in the case of artin algebras. The relevant example goes back to Igusa, Smalø, and Todorov, [54]:

Example 5.12. Let k be an algebraically closed field and R the finite dimensional monomial algebra given by the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left=50, "\gamma"] & 2 \arrow[l, bend left=30, "\beta"'] \arrow[l, bend left=70, "\alpha"] \end{tikzcd}$$

with the relations $\alpha\gamma = \beta\gamma = \gamma\alpha = 0$.
Then $\mathrm{Fin\,dim}\, R = \mathrm{fin\,dim}\, R = 1$, so $\varinjlim \mathcal{P}^{<\omega} = \mathcal{P}$.
However, using the fact that R has a factor algebra isomorphic to the Kronecker algebra Λ, and the representation theory of Λ–modules developed by Ringel, one can show that $\mathcal{A}_f \neq \mathcal{P}$, that is, $\mathcal{A}_f$ is not closed under direct limits. For more details, we refer to [8].

The module T_f is not finite–dimensional. This follows from another important property of this example proved in [54], namely that $\mathcal{P}^{<\omega}$ is not contravariantly finite, and from Theorem 5.19 below.

Before proving the finitistic dimension conjectures for all artin algebras with $\mathcal{P}^{<\omega}$ contravariantly finite, we recall a formula for computing $\mathrm{fin\,dim}\, R$ for right artinian rings proved in [74]:

Theorem 5.13. *Let R be a right artinian ring. For each $S \in \mathrm{simp}\, R$, take a special $\mathcal{A}_f$–precover, $f_S : X_S \to S$. Then*

$$\mathrm{fin\,dim}\, R = \max\{\mathrm{proj\,dim}\, X_S \mid S \in \mathrm{simp}\, R\}.$$

In particular, $\mathrm{fin\,dim}\, R < \infty$, *iff $X_S \in \mathcal{P}$ for each $S \in \mathrm{simp}\, R$.*

Finally, we will employ infinite–dimensional tilting theory in the proof of the second finitistic dimension conjecture for right artinian rings such that $\mathcal{P}^{<\omega}$ is contravariantly finite, and (ii) in the proof of the first finitistic dimension conjecture for artin algebras with $\mathcal{P}^{<\omega}$ contravariantly finite.

The first result was proved in case of artin algebras by Auslander and Reiten [13], the second by Huisgen–Zimmermann and Smalø [53]. Our approach is different, based on [7] and [74].

An class of artinian rings where $\mathcal{P}^{<\omega}$ is contravariantly finite was studied by Huisgen–Zimmermann:

Example 5.14. Let R be a right artinian right serial ring (that is, $R = \bigoplus_{i<m} e_i R$, where $e_i R$ is a right artinian right uniserial module and e_i is a primitive idempotent for each $i < m$). Then $\mathcal{P}^{<\omega}$ is contravariantly finite. Moreover, $\operatorname{fin\,dim} R = 1 + \max\{\operatorname{proj\,dim} e_i J^l \mid l < n, i < m, \operatorname{proj\,dim} e_i J^l < \infty\}$. Here $J = \operatorname{Rad}(R)$ is the Jacobson radical of R, and n the nilpotency index of J. For more details, we refer to [52].

The following is a criterion for contravariant finiteness of $\mathcal{P}^{<\omega}$ in terms of the $\mathcal{A}_f$–approximations of simple modules:

Theorem 5.15. *Let R be a right artinian ring. Then $\mathcal{P}^{<\omega}$ is contravariantly finite, iff we can choose $X_S \in \text{mod–}R$ for all $S \in \operatorname{simp} R$.*

Proof. Assume $X_S \in \text{mod–}R$ for all $S \in \operatorname{simp} R$. Then each finitely generated module F has a special $\mathcal{A}_f$–precover $X_F \to F$ such that X_F is finitely $\mathcal{C}$–filtered. Hence $X_F \in \mathcal{P}^{<\omega}$, and $\mathcal{P}^{<\omega}$ is contravariantly finite.

Conversely, let $g_S : Y_S \to S$ be a $\mathcal{P}^{<\omega}$–precover of S in mod–R. By Corollary 1.9, we can w.l.o.g. assume that g_S is a $\mathcal{P}^{<\omega}$–cover. By a version of Lemma 1.12 in mod–R, $\operatorname{Ker}(g_S) \in (\mathcal{P}^{<\omega})^{\perp}$. So g_S is a special $\mathcal{A}_f$–precover of S. ∎

As a corollary we obtain a sufficient condition for finiteness of the little finitistic dimension of right artinian rings:

Corollary 5.16. *Let R be a right artinian ring. If $\mathcal{P}^{<\omega}$ is contravariantly finite, then $\operatorname{fin\,dim} R < \infty$.*

Proof. By Theorem 5.15, each simple module S has an $\mathcal{A}_f$–approximation $X_S \to S$ such that $X_S \in \mathcal{P}^{<\omega}$. By Theorem 5.13, $\operatorname{fin\,dim} R < \infty$. ∎

The sufficient condition of Corollary 5.16 is not necessary even in the case when A is a finite–dimensional monomial algebra over an algebraically closed field: the IST–algebra defined in Example 5.12 above satisfies $\operatorname{fin\,dim} R = \operatorname{Fin\,dim} R = 1$, but $\mathcal{P}^{<\omega}$ is not contravariantly finite.

Using tilting approximations, we will now prove that contravariant finiteness of $\mathcal{P}^{<\omega}$ for an artin algebra also implies the validity of the first finitistic dimension conjecture. First we will need a lemma making use of an idea of Auslander and Buchweiz [12]:

Lemma 5.17. *Let R be an artin algebra over a commutative artinian ring k, and T be a finitely generated tilting module of projective dimension n. Let $\mathcal{C} = T^{\perp_\infty} \cap \text{mod–}R$.*

(a) *Each module $M \in \text{mod–}R$ has $\mathcal{C}$–coresolution dimension $\leq n$.*

(b) *The class $\mathcal{C}$ is covariantly finite, and $\mathcal{D} = {}^{\perp}\mathcal{C} \cap \text{mod–}R$ is contravariantly finite.*

Proof. (a) The claim is trivial for $n = 0$, since in this case $\mathcal{C} = \text{mod--}R$. For $n > 0$, consider the long exact sequence

$$0 \to M \to I_0 \to \ldots \to I_{n-1} \to N \to 0,$$

where all I_i ($i < n$) are finitely generated injective, hence $I_i \in \mathcal{C}$.
Since $\mathrm{Ext}_R^{m+n}(T, M) \cong \mathrm{Ext}_R^m(T, N) = 0$ for all $m \geq 1$, also $N \in \mathcal{C}$.

(b) Let $M \in \text{mod--}R$. Let $m < \omega$ denote the $\mathcal{C}$–coresolution dimension of M. By induction on m, we prove that there are two short exact sequences $0 \to V_1 \to U_1 \xrightarrow{\varphi_1} M \to 0$ and $0 \to M \xrightarrow{\varphi_2} V_2 \to U_2 \to 0$ such that U_1 and U_2 have finite $\mathrm{add}(T)$–coresolution dimension and $V_1, V_2 \in \mathcal{C}$.

This is sufficient: since $\mathcal{D}$ is resolving in mod–R and $\mathrm{add}(T) \subseteq \mathcal{D}$, we have $U_1, U_2 \in \mathcal{D}$. So φ_1 (φ_2) is a special $\mathcal{D}$–precover ($\mathcal{C}$–preenvelope) of the module M in mod–R.

Assume $m = 0$, so $M \in \mathcal{C}$. For the first short exact sequence, we take $0 \to M \to M \to 0 \to 0$. For the second, we take an exact sequence $0 \to C \to Q \to M \to 0$ with $Q \in \mathrm{add}(T)$ and $C \in \mathcal{C}$.

The latter sequence exists, as the k–module $\mathrm{Hom}_R(T, M)$ has a finite k–generating set S, so the canonical map $f : T^{(S)} \to M$ has the property that any $g \in \mathrm{Hom}_R(T, M)$ can be factorized through f. Since $\mathcal{C} \subseteq \mathrm{Gen}(T)$ by Lemma 3.8 (b), f is surjective, and we put $C = \mathrm{Ker}(f)$ and $Q = T^{(S)}$. This is possible since $C \in T^{\perp_\infty}$: indeed, $\mathrm{Ext}_R^1(T, C) = 0$, because $\mathrm{Hom}_R(T, f)$ is surjective by construction and T is tilting; moreover, dimension shifting gives $\mathrm{Ext}_R^{i+1}(T, C) = \mathrm{Ext}_R^i(T, M) = 0$ for each $i \geq 1$, since $M \in \mathcal{C}$.

For the inductive step we split a $\mathcal{C}$–coresolution of M of length $m + 1$ into two parts: the exact sequence $0 \to M \to C_0 \xrightarrow{\pi} K \to 0$, and the long exact sequence of length m:

$$0 \to K \to C_1 \to \ldots \to C_m \to 0.$$

By inductive premise, there is an exact sequence $0 \to C \to Q \xrightarrow{\rho} K \to 0$ with Q of finite $\mathrm{add}(T)$–coresolution dimension and $C \in \mathcal{C}$. Consider the pullback of π and ρ:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & C \arrow[r, equal] \arrow[d] & C \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "\sigma"] \arrow[d, equal] & P \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, "\rho"'] & 0 \\
0 \arrow[r] & M \arrow[r] & C_0 \arrow[r, "\pi"] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

In the middle row, the module Q is finitely $\mathrm{add}(T)$–coresolved and $P \in \mathcal{C}$, so the row provides for the second short exact sequence for M.

Since $P \in \mathcal{C}$, as above, we obtain an exact sequence $0 \to V \to U' \xrightarrow{\tau} P \to 0$ with $U' \in \operatorname{add}(T)$ and $V \in \mathcal{C}$. Consider the pullback of σ and τ:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & V \arrow[r, equal] \arrow[d] & V \arrow[d] & & \\
0 \arrow[r] & U \arrow[r] \arrow[d] & U' \arrow[r] \arrow[d, "\tau"'] & Q \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "\sigma"] \arrow[d] & P \arrow[r] \arrow[d] & Q \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

The middle row shows that U has a finite $\operatorname{add}(T)$–coresolution. So the left column provides for the first short exact sequence required for M. ∎

The following lemma gives the connection between contravariant finiteness and finite number of generators of the tilting module:

Lemma 5.18. *Let R be an artin algebra. Let $\mathcal{S}$ be a syzygy closed subclass of $\mathcal{P}^{<\omega}$. Denote by $(\mathcal{U}, \mathcal{V})$ the cotorsion pair generated by $\mathcal{S}$. Then the following assertions are equivalent:*

(a) *$\mathcal{U}^{<\omega}$ is contravariantly finite.*

(b) *There exists a finitely generated tilting module T such that $\mathcal{V} = T^{\perp_\infty}$.*

Proof. (a) implies (b): first $\mathcal{U}^{<\omega} \subseteq \mathcal{A}_f^{<\omega} = \mathcal{P}^{<\omega}$. Let $g_S : U_S \to S$ denote a $\mathcal{U}^{<\omega}$–precover of a simple module $S \in \operatorname{simp} R$. By Corollary 1.9, we can w.l.o.g. assume that g_S is a $\mathcal{U}^{<\omega}$–cover. By a version of Lemma 1.12 in mod–R, $\operatorname{Ker}(g_S) \in (\mathcal{U}^{<\omega})^\perp = \mathcal{V}$. So g_S is a $\mathcal{U}$–cover of S.

Let $n = \max\{\operatorname{proj\,dim} U_S \mid S \in \operatorname{simp} R\}$. Since $\operatorname{simp} R$ is a finite set, we have $n < \infty$, and $\mathcal{S} \subseteq \mathcal{U} \cap \text{mod–}R \subseteq \mathcal{P}_n^{<\omega}$. So $\mathcal{U} \subseteq \mathcal{P}_n$, and Lemma 5.6 provides an n–tilting module T' such that $\mathcal{V} = (T')^{\perp_\infty}$. Moreover, T' is equivalent to the tilting module $T = \bigoplus_{i \leq n} T_i$, where T_0 is any special $\mathcal{V}$–preenvelope of R, T_1, any special $\mathcal{V}$–preenvelope of T_0/R etc.

Since each finitely generated module X has a special $\mathcal{U}$–precover $g_X : U_X \to X$ such that U_X is finitely $\{U_S \mid S \in \operatorname{simp} R\}$–filtered, each finitely generated module X has a special $\mathcal{V}$–preenvelope $f_X : X \to V_X$ with $V_X \in \text{mod–}R$.
It follows that all T_i $(i \leq n)$, and hence T, can be taken finitely generated.

(b) implies (a): by (the proof of) Lemma 5.17, for any module $Y \in \text{mod–}R$ there is an exact sequence $0 \to V \to U \to Y \to 0$ such that $V \in \mathcal{V}^{<\omega}$ and U has a finite $\operatorname{add}(T)$–coresolution. Hence $U \in \mathcal{U}^{<\omega}$, and $\mathcal{U}^{<\omega}$ is contravariantly finite. ∎

Now we can characterize the artin algebras with $\mathcal{P}^{<\omega}$ contravariantly finite:

Theorem 5.19. *Let R be an artin algebra. The following assertions are equivalent:*

(a) $\mathcal{P}^{<\omega}$ *is contravariantly finite.*

(b) *There is a finitely generated tilting module T_f such that $\mathcal{B}_f = T_f^{\perp_\infty}$.*

Proof. Since $\mathcal{P}^{<\omega} = \mathcal{A}_f \cap \text{mod–}R$. So the assertion follows from Lemma 5.18 for $\mathcal{S} = \mathcal{P}^{<\omega}$. ∎

The following lemma is crucial:

Lemma 5.20. *Let R be a ring, T a tilting module, and $(\mathcal{A}, \mathcal{B})$ the tilting cotorsion pair induced by T. Then the following conditions (a) and (b) are equivalent:*

(a) $\mathrm{Add}(T)$ *is closed under cokernels of monomorphisms.*

(b) $\mathcal{A} = \mathcal{P}$.

These conditions imply $\operatorname{Fin\,dim} R < \infty$. *If $T \in \text{mod–}R$, then these conditions also imply*

(c) $\mathcal{A} = \mathcal{A}_f$.

If $T \in \text{mod–}R$ is Σ–pure–injective, then condition (c) implies (a), so all the three conditions are equivalent. In this case $\operatorname{fin\,dim} R = \operatorname{Fin\,dim} R < \infty$.

Proof. Assume (a). On the one hand, by Lemma 3.8 (b), $\mathcal{A} \subseteq \mathcal{P}_n \subseteq \mathcal{P}$, where n is the projective dimension of T.

On the other hand, if $M \in \mathcal{P}$, then the completeness of $(\mathcal{A}, \mathcal{B})$ yields an exact sequence $0 \to M \to B \to A \to 0$ with $B \in \mathcal{B}$ and $A \in \mathcal{A}$. Clearly $B \in \mathcal{P}$. Let $m = \operatorname{proj\,dim} B$. By Lemma 3.8 (d), there is a long exact sequence

$$0 \to M_m \to \cdots \to M_0 \to B \to 0$$

with $M_i \in \mathrm{Add}(T)$ for all $i \leq m$. Using the assumption, and induction, we obtain that $B \in \mathrm{Add}(T) \subseteq \mathcal{A}$. Since $\mathcal{A}$ is resolving, we get $M \in \mathcal{A}$. This proves that $\mathcal{P} \subseteq \mathcal{A}$, and hence $\mathcal{P} = \mathcal{A}$.
Assume (b). Then $\operatorname{Add} T = \mathcal{P} \cap \mathcal{B}$ by Lemma 3.8 (c). So $\operatorname{Add} T$ is closed under cokernels of monomorphisms, since $\mathcal{P}$ and $\mathcal{B}$ share this property.
Moreover, (b) implies that $\mathcal{P}$ is closed under direct sums, hence there is $n < \omega$ with $\mathcal{P} = \mathcal{P}_n$, so $\operatorname{Fin\,dim} R < \infty$. Assume $T \in \text{mod–}R$. We will show that (b) implies (c): indeed, we have $\mathcal{A}_f \subseteq \mathcal{A} = \mathcal{P} = \mathcal{P}_n$. By assumption, $T \in \mathcal{A}_f$, so $\mathcal{A} \subseteq A_f$.
Now assume that $T \in \text{mod–}R$ is Σ–pure–injective. We will prove that (c) implies (a):

First we show that every monomorphism in $\mathrm{add}(T)$ splits. Indeed, let $0 \to A \xrightarrow{f} B \to C \to 0$ be a short exact sequence with A and B in $\mathrm{add}(T)$. By assumption, $\mathrm{add}(T) = \text{mod–}R \cap \mathrm{Add}(T) \subseteq \mathcal{B}_f \cap \mathcal{P}^{<\omega}$. Since $\mathcal{P}^{<\omega}$ is closed under cokernels of monomorphisms, we have $A \in \mathcal{B}_f$ and $C \in \mathcal{P}^{<\omega} \subseteq \mathcal{A}_f$. Thus $\mathrm{Ext}^1_R(C, A) = 0$ and f splits.

Now let $A \subseteq B$ and $A, B \in \mathrm{Add}(T)$. By assumption, each element of $\mathrm{Add}(T)$ is isomorphic to a direct sum of finitely generated direct summands of T. Let $\sum_{j \in J} x_j r_{ij} = a_i$ $(i \in I)$ be a finite system of R–linear equations with $a_i \in A$ $(i \in I)$ which is solvable in B, by $x_j = b_j$ $(j \in J)$. There is a finitely generated direct summand $A' \subseteq A$ such that $a_i \in A'$ for all $i \in I$, and a finitely generated direct summand $B' \subseteq B$ such that $A' \subseteq B'$ and $b_j \in B'$ for all $j \in J$. By the previous paragraph, the embedding $A' \subseteq B'$ is pure (even split), so the finite system is also solvable by some $x_j = a'_j \in A'$ $(j \in J)$.

This proves that any monomorphism in $\mathrm{Add}(T)$ is pure. Since T is Σ–pure–injective, each monomorphism in $\mathrm{Add}(T)$ splits, and (a) holds.

Finally, (b) and (c) give $\mathcal{A}_f = \mathcal{P}$, so each module of finite projective dimension is a direct summand of a $\mathcal{P}^{<\omega}$–filtered module. It follows that $\mathrm{fin\,dim} R = \mathrm{Fin\,dim}\, R$. ∎

Now we can prove the main result of this section:

Theorem 5.21. *Assume that R is an artin algebra such that $\mathcal{P}^{<\omega}$ is contravariantly finite. Then every module of finite projective dimension is a direct summand of a $\mathcal{P}^{<\omega}$–filtered module, and* $\mathrm{Fin\,dim}\, R = \mathrm{fin\,dim}\, R < \infty$.

Proof. By Theorem 5.19, there is a finitely generated tilting module T_f such that $\mathcal{B}_f = T_f^{\perp_\infty}$. Clearly T_f is Σ–pure–injective and condition (c) of Lemma 5.20 holds for $T = T_f$. Condition (b) then gives $\mathcal{A}_f = \mathcal{P}$, and the final claim follows again by Lemma 5.20. ∎

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