Complex variable
solved problems

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Contents

Acknowledgement. The following problems were solved using my own procedure in a program Maple V, release 5. All possible errors are my faults.

1 Residue theorem problems

We will solve several problems using the following theorem:

Theorem. (Residue theorem) Suppose U is a simply connected open subset of the complex plane, and $w_1, \ldots, w_n$ are finitely many points of U and f is a function which is defined and holomorphic on $U \setminus \{w_1, \ldots, w_n\}$. If φ is a simply closed curve in U contaning the points w_k in the interior, then

$$\oint_{\varphi} f(z)\, dz = 2\pi i \sum_{k=1}^{k} \operatorname{res}(f, w_k)\,.$$

The following rules can be used for residue counting:

Theorem. (Rule 1) If f has a pole of order k at the point w then

$$\operatorname{res}(f, w) = \frac{1}{(k-1)!} \lim_{z \to w} \left((z-w)^k f(z)\right)^{(k-1)}.$$

Theorem. (Rule 2) If f, g are holomorphic at the point w and $f(w) \neq 0$. If $g(w) = 0, g'(w) \neq 0$, then

$$\operatorname{res}\left(\frac{f}{g}, w\right) = \frac{f(w)}{g'(w)}\,.$$

Theorem. (Rule 3) If h is holomorphic at w and g has a simple pole at w, then $\operatorname{res}(gh, w) = h(w) \operatorname{res}(g, w)$.

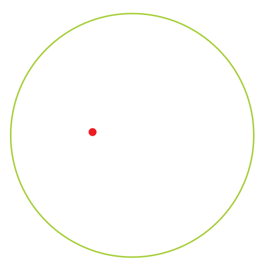
We will use special formulas for special types of problems:

Theorem. (TYPE I. Integral from a rational function in sin and cos.) If $Q(a,b)$ is a rational function of two complex variables such that for real a,b, $a^2+b^2=1$ is $Q(a,b)$ finite, then the function

$$T(z) := Q\left(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\right)/(iz)$$

is rational, has no poles on the real line and

$$\int_0^{2\pi} Q(\cos t, \sin t)\, dt = 2\pi i \cdot \sum_{|a|<1,\, T(a)=\infty} \operatorname{res}(T,a)\,.$$



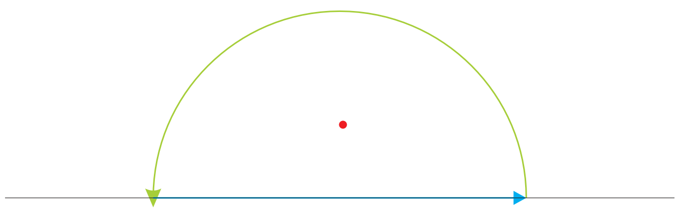
Example

$$\int_0^{2\,\pi} \frac{1}{2+\cos(x)}\, dx$$

Theorem. (TYPE II. Integral from a rational function.) Suppose P, Q are polynomial of order m, n respectively, and $n - m > 1$ and Q has no real roots. Then for the rational function $f = \frac{P}{Q}$ holds

$$\int_{-\infty}^{+\infty} f(x)\, dx = 2\pi i \sum_k \operatorname{res}\left(f, w_k\right),$$

where all singularities of f with a positive imaginary part are considered in the above sum.



Example

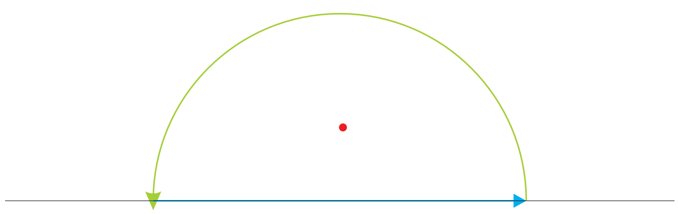
$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx$$

Theorem. (TYPE III. Integral from a rational function multiplied by cos or sin) If Q is a rational function such that has no pole at the real line and for $z \to \infty$ is $Q(z) = O(z^{-1})$. For $b > 0$ denote $f(z) = Q(z)\, e^{ibz}$. Then

$$\int_{-\infty}^{+\infty} Q(x)\cos(bx)\, dx = \mathrm{Re}\left(2\pi i \cdot \sum_{w} \mathrm{res}\,(f, w)\right)$$

$$\int_{-\infty}^{+\infty} Q(x)\sin(bx)\, dx = \mathrm{Im}\left(2\pi i \cdot \sum_{w} \mathrm{res}\,(f, w)\right)$$

where only w with a positive imaginary part are considered in the above sums.



Example

$$\int_{-\infty}^{+\infty} \frac{\cos(x)}{1+x^2}\, dx$$

1.1 Problem.

Using the Residue theorem evaluate

$$\int_0^{2\pi} \frac{\cos(x)^2}{13 + 12\cos(x)}\, dx$$

Hint.

$$\mathit{Type\_I}$$

Solution.
We denote

$$\mathrm{f}(z) = -\frac{I\,(\frac{1}{2}\,z + \frac{1}{2}\,\frac{1}{z})^2}{(13 + 6\,z + 6\,\frac{1}{z})\,z}$$

We find singularities

$$[\{z = 0\},\, \{z = \frac{-3}{2}\},\, \{z = \frac{-2}{3}\}]$$

The singularity

$$z = 0$$

is in our region and we will add the following residue

$$\mathrm{res}(0,\, \mathrm{f}(z)) = \frac{13}{144}\,I$$

The singularity

$$z = \frac{-3}{2}$$

will be skipped because the singularity is not in our region.
The singularity

$$z = \frac{-2}{3}$$

is in our region and we will add the following residue

$$\mathrm{res}(\frac{-2}{3},\, \mathrm{f}(z)) = -\frac{169}{720}\,I$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\, \mathrm{f}(z))) = \frac{13}{45}\,\pi$$

The solution is

$$\int_0^{2\pi} \frac{\cos(x)^2}{13 + 12\cos(x)}\, dx = \frac{13}{45}\,\pi$$

We can try to solve it using real calculus and obtain the result

$$\int_0^{2\,\pi} \frac{\cos(x)^2}{13 + 12\cos(x)}\,dx = \frac{13}{45}\,\pi$$

Info.

not_given

Comment.

no_comment

1.2 Problem.

Using the Residue theorem evaluate

$$\int_0^{2\,\pi} \frac{\cos(x)^4}{1+\sin(x)^2}\,dx$$

Hint.

$$\mathit{Type\_I}$$

Solution.
We denote

$$\mathrm{f}(z) = -\frac{I\,(\frac{1}{2}\,z + \frac{1}{2}\,\frac{1}{z})^4}{(1 - \frac{1}{4}\,(z - \frac{1}{z})^2)\,z}$$

We find singularities

$[\{z = 0\}, \{z = \sqrt{2}+1\}, \{z = 1-\sqrt{2}\}, \{z = \sqrt{2}-1\}, \{z = -1-\sqrt{2}\}, \{z = \infty\}, \{z = -\infty\}]$
The singularity

$$z = 0$$

is in our region and we will add the following residue

$$\mathrm{res}(0,\,\mathrm{f}(z)) = \frac{5}{2}\,I$$

The singularity

$$z = \sqrt{2} + 1$$

will be skipped because the singularity is not in our region.
The singularity

$$z = 1 - \sqrt{2}$$

is in our region and we will add the following residue

$$\mathrm{res}(1 - \sqrt{2},\,\mathrm{f}(z)) = \frac{-768\,I\,\sqrt{2} + 1088\,I}{768 - 544\,\sqrt{2}}$$

The singularity

$$z = \sqrt{2} - 1$$

is in our region and we will add the following residue

$$\mathrm{res}(\sqrt{2} - 1,\,\mathrm{f}(z)) = \frac{-768\,I\,\sqrt{2} + 1088\,I}{768 - 544\,\sqrt{2}}$$

The singularity

$$z = -1 - \sqrt{2}$$

will be skipped because the singularity is not in our region.
The singularity

$$z = \infty$$

will be skipped because only residues at finite singularities are counted.
The singularity

$$z = -\infty$$

will be skipped because only residues at finite singularities are counted.

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\,\mathrm{f}(z))) = 2\,I\,\pi\,(\frac{5}{2}\,I + 2\,\frac{-768\,I\,\sqrt{2} + 1088\,I}{768 - 544\,\sqrt{2}})$$

The solution is

$$\int_0^{2\,\pi} \frac{\cos(x)^4}{1 + \sin(x)^2}\,dx = 2\,I\,\pi\,(\frac{5}{2}\,I + 2\,\frac{-768\,I\,\sqrt{2} + 1088\,I}{768 - 544\,\sqrt{2}})$$

We can try to solve it using real calculus and obtain the result

$$\int_0^{2\,\pi} \frac{\cos(x)^4}{1 + \sin(x)^2}\,dx = \frac{\pi\,(4\,\sqrt{2} - 5)}{(\sqrt{2} + 1)\,(\sqrt{2} - 1)}$$

Info.

$$\mathit{not\_given}$$

Comment.

$$\mathit{no\_comment}$$

1.3 Problem.

Using the Residue theorem evaluate

$$\int_0^{2\pi} \frac{\sin(x)^2}{\frac{5}{4} - \cos(x)}\, dx$$

Hint.

$$\mathit{Type\_I}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{1}{4}\, \frac{I\,(z - \frac{1}{z})^2}{(\frac{5}{4} - \frac{1}{2}\,z - \frac{1}{2}\,\frac{1}{z})\,z}$$

We find singularities

$$[\{z = 0\},\, \{z = \frac{1}{2}\},\, \{z = 2\}]$$

The singularity

$$z = 0$$

is in our region and we will add the following residue

$$\mathrm{res}(0,\, \mathrm{f}(z)) = -\frac{5}{4}\, I$$

The singularity

$$z = \frac{1}{2}$$

is in our region and we will add the following residue

$$\mathrm{res}(\frac{1}{2},\, \mathrm{f}(z)) = \frac{3}{4}\, I$$

The singularity

$$z = 2$$

will be skipped because the singularity is not in our region.

Our sum is

$$2\, I\, \pi\, (\sum \mathrm{res}(z,\, \mathrm{f}(z))) = \pi$$

The solution is

$$\int_0^{2\,\pi} \frac{\sin(x)^2}{\frac{5}{4} - \cos(x)}\,dx = \pi$$

We can try to solve it using real calculus and obtain the result

$$\int_0^{2\,\pi} \frac{\sin(x)^2}{\frac{5}{4} - \cos(x)}\,dx = \pi$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

1.4 Problem.

Using the Residue theorem evaluate

$$\int_0^{2\pi} \frac{1}{\sin(x)^2 + 4\cos(x)^2}\,dx$$

Hint.

$$\mathit{Type\_I}$$

Solution.
We denote

$$\mathrm{f}(z) = -\frac{I}{(-\frac{1}{4}(z - \frac{1}{z})^2 + 4(\frac{1}{2}z + \frac{1}{2}\frac{1}{z})^2)\,z}$$

We find singularities

$$[\{z = -\frac{1}{3} I \sqrt{3}\}, \{z = \frac{1}{3} I \sqrt{3}\}, \{z = I\sqrt{3}\}, \{z = -I\sqrt{3}\}]$$

The singularity

$$z = -\frac{1}{3} I \sqrt{3}$$

is in our region and we will add the following residue

$$\mathrm{res}(-\frac{1}{3} I \sqrt{3}, \mathrm{f}(z)) = -\frac{1}{4} I$$

The singularity

$$z = \frac{1}{3} I \sqrt{3}$$

is in our region and we will add the following residue

$$\mathrm{res}(\frac{1}{3} I \sqrt{3}, \mathrm{f}(z)) = -\frac{1}{4} I$$

The singularity

$$z = I\sqrt{3}$$

will be skipped because the singularity is not in our region.
The singularity

$$z = -I\sqrt{3}$$

will be skipped because the singularity is not in our region.

Our sum is

$$2 I \pi (\sum \mathrm{res}(z, \mathrm{f}(z))) = \pi$$

The solution is

$$\int_0^{2\pi} \frac{1}{\sin(x)^2 + 4\cos(x)^2}\, dx = \pi$$

We can try to solve it using real calculus and obtain the result

$$\int_0^{2\pi} \frac{1}{\sin(x)^2 + 4\cos(x)^2}\, dx = \pi$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

1.5 Problem.

Using the Residue theorem evaluate

$$\int_0^{2\,\pi} \frac{1}{13 + 12\sin(x)}\,dx$$

Hint.

$$\mathit{Type\_I}$$

Solution.
We denote

$$\mathrm{f}(z) = -\frac{I}{(13 - 6\,I\,(z - \frac{1}{z}))\,z}$$

We find singularities

$$[\{z = -\frac{2}{3}\,I\},\, \{z = -\frac{3}{2}\,I\}]$$

The singularity

$$z = -\frac{2}{3}\,I$$

is in our region and we will add the following residue

$$\mathrm{res}(-\frac{2}{3}\,I,\, \mathrm{f}(z)) = -\frac{1}{5}\,I$$

The singularity

$$z = -\frac{3}{2}\,I$$

will be skipped because the singularity is not in our region.

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\, \mathrm{f}(z))) = \frac{2}{5}\,\pi$$

The solution is

$$\int_0^{2\,\pi} \frac{1}{13 + 12\sin(x)}\,dx = \frac{2}{5}\,\pi$$

We can try to solve it using real calculus and obtain the result

$$\int_0^{2\,\pi} \frac{1}{13 + 12\sin(x)}\,dx = \frac{2}{5}\,\pi$$

Info.

$$\mathit{not\_given}$$

Comment.

$$no\_comment$$

1.6 Problem.

Using the Residue theorem evaluate

$$\int_0^{2\pi} \frac{1}{2+\cos(x)}\,dx$$

Hint.

$$\mathit{Type\_I}$$

Solution.
We denote

$$\mathrm{f}(z) = -\frac{I}{(2+\frac{1}{2}\,z+\frac{1}{2}\,\frac{1}{z})\,z}$$

We find singularities

$$[\{z=-2+\sqrt{3}\},\,\{z=-2-\sqrt{3}\}]$$

The singularity

$$z=-2+\sqrt{3}$$

is in our region and we will add the following residue

$$\mathrm{res}(-2+\sqrt{3},\,\mathrm{f}(z)) = -\frac{1}{3}\,I\,\sqrt{3}$$

The singularity

$$z=-2-\sqrt{3}$$

will be skipped because the singularity is not in our region.

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\,\mathrm{f}(z))) = \frac{2}{3}\,\pi\,\sqrt{3}$$

The solution is

$$\int_0^{2\pi} \frac{1}{2+\cos(x)}\,dx = \frac{2}{3}\,\pi\,\sqrt{3}$$

We can try to solve it using real calculus and obtain the result

$$\int_0^{2\pi} \frac{1}{2+\cos(x)}\,dx = \frac{2}{3}\,\pi\,\sqrt{3}$$

Info.

$$\mathit{not\_given}$$

Comment.

$$\mathit{no\_comment}$$

1.7 Problem.

Using the Residue theorem evaluate

$$\int_{0}^{2\,\pi} \frac{1}{(2+\cos(x))^2}\,dx$$

Hint.

$$\mathit{Type\_I}$$

Solution.
We denote

$$\mathrm{f}(z) = -\frac{I}{(2+\frac{1}{2}\,z+\frac{1}{2}\,\frac{1}{z})^2\,z}$$

We find singularities

$$[\{z=-2+\sqrt{3}\},\,\{z=-2-\sqrt{3}\}]$$

The singularity

$$z=-2+\sqrt{3}$$

is in our region and we will add the following residue

$$\mathrm{res}(-2+\sqrt{3},\,\mathrm{f}(z)) = -\frac{1}{3}\,I+\frac{1}{3}\,(-\frac{2}{3}\,I+\frac{1}{3}\,I\,\sqrt{3})\,\sqrt{3}$$

The singularity

$$z=-2-\sqrt{3}$$

will be skipped because the singularity is not in our region.

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\,\mathrm{f}(z))) = 2\,I\,\pi\,(-\frac{1}{3}\,I+\frac{1}{3}\,(-\frac{2}{3}\,I+\frac{1}{3}\,I\,\sqrt{3})\,\sqrt{3})$$

The solution is

$$\int_{0}^{2\,\pi} \frac{1}{(2+\cos(x))^2}\,dx = 2\,I\,\pi\,(-\frac{1}{3}\,I+\frac{1}{3}\,(-\frac{2}{3}\,I+\frac{1}{3}\,I\,\sqrt{3})\,\sqrt{3})$$

We can try to solve it using real calculus and obtain the result

$$\int_{0}^{2\,\pi} \frac{1}{(2+\cos(x))^2}\,dx = \frac{4}{9}\,\pi\,\sqrt{3}$$

Info.

$$\mathit{not\_given}$$

Comment.

$$\mathit{no\_comment}$$

1.8 Problem.

Using the Residue theorem evaluate

$$\int_0^{2\,\pi} \frac{1}{2+\sin(x)}\,dx$$

Hint.

$$\textit{Type\_I}$$

Solution.
We denote

$$\mathrm{f}(z) = -\frac{I}{(2-\frac{1}{2}\,I\,(z-\frac{1}{z}))\,z}$$

We find singularities

$$[\{z = -2\,I + I\,\sqrt{3}\},\, \{z = -2\,I - I\,\sqrt{3}\}]$$

The singularity

$$z = -2\,I + I\,\sqrt{3}$$

is in our region and we will add the following residue

$$\mathrm{res}(-2\,I + I\,\sqrt{3},\, \mathrm{f}(z)) = -\frac{1}{3}\,I\,\sqrt{3}$$

The singularity

$$z = -2\,I - I\,\sqrt{3}$$

will be skipped because the singularity is not in our region.

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\, \mathrm{f}(z))) = \frac{2}{3}\,\pi\,\sqrt{3}$$

The solution is

$$\int_0^{2\,\pi} \frac{1}{2+\sin(x)}\,dx = \frac{2}{3}\,\pi\,\sqrt{3}$$

We can try to solve it using real calculus and obtain the result

$$\int_0^{2\,\pi} \frac{1}{2+\sin(x)}\,dx = \frac{2}{3}\,\pi\,\sqrt{3}$$

Info.

$$\textit{not\_given}$$

Comment.

$$\textit{no\_comment}$$

1.9 Problem.

Using the Residue theorem evaluate

$$\int_0^{2\,\pi} \frac{1}{5 + 4\cos(x)}\,dx$$

Hint.

$$\mathit{Type\_I}$$

Solution.
We denote

$$\mathrm{f}(z) = -\frac{I}{(5 + 2\,z + 2\,\frac{1}{z})\,z}$$

We find singularities

$$[\{z = -2\},\ \{z = \frac{-1}{2}\}]$$

The singularity

$$z = -2$$

will be skipped because the singularity is not in our region.
The singularity

$$z = \frac{-1}{2}$$

is in our region and we will add the following residue

$$\mathrm{res}(\frac{-1}{2},\ \mathrm{f}(z)) = -\frac{1}{3}\,I$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\ \mathrm{f}(z))) = \frac{2}{3}\,\pi$$

The solution is

$$\int_0^{2\,\pi} \frac{1}{5 + 4\cos(x)}\,dx = \frac{2}{3}\,\pi$$

We can try to solve it using real calculus and obtain the result

$$\int_0^{2\,\pi} \frac{1}{5 + 4\cos(x)}\,dx = \frac{2}{3}\,\pi$$

Info.

$$\mathit{not\_given}$$

Comment.

$$\mathit{no\_comment}$$

1.10 Problem.

Using the Residue theorem evaluate

$$\int_0^{2\pi} \frac{1}{5 + 4\sin(x)}\, dx$$

Hint.

$$\mathit{Type\_I}$$

Solution.
We denote

$$\mathrm{f}(z) = -\frac{I}{(5 - 2\, I\, (z - \frac{1}{z}))\, z}$$

We find singularities

$$[\{z = -2\, I\},\, \{z = -\frac{1}{2}\, I\}]$$

The singularity

$$z = -2\, I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = -\frac{1}{2}\, I$$

is in our region and we will add the following residue

$$\mathrm{res}(-\frac{1}{2}\, I,\, \mathrm{f}(z)) = -\frac{1}{3}\, I$$

Our sum is

$$2\, I\, \pi\, (\sum \mathrm{res}(z,\, \mathrm{f}(z))) = \frac{2}{3}\, \pi$$

The solution is

$$\int_0^{2\pi} \frac{1}{5 + 4\sin(x)}\, dx = \frac{2}{3}\, \pi$$

We can try to solve it using real calculus and obtain the result

$$\int_0^{2\pi} \frac{1}{5 + 4\sin(x)}\, dx = \frac{2}{3}\, \pi$$

Info.

$$\mathit{not\_given}$$

Comment.

$$\mathit{no\_comment}$$

1.11 Problem.

Using the Residue theorem evaluate

$$\int_0^{2\pi} \frac{\cos(x)}{\frac{5}{4} - \cos(x)} \, dx$$

Hint.

$$\mathit{Type\_I}$$

Solution.
We denote

$$\mathrm{f}(z) = -\frac{I\,(\frac{1}{2}\,z + \frac{1}{2}\,\frac{1}{z})}{(\frac{5}{4} - \frac{1}{2}\,z - \frac{1}{2}\,\frac{1}{z})\,z}$$

We find singularities

$$[\{z = 0\},\, \{z = \frac{1}{2}\},\, \{z = 2\}]$$

The singularity

$$z = 0$$

is in our region and we will add the following residue

$$\mathrm{res}(0,\, \mathrm{f}(z)) = I$$

The singularity

$$z = \frac{1}{2}$$

is in our region and we will add the following residue

$$\mathrm{res}(\frac{1}{2},\, \mathrm{f}(z)) = -\frac{5}{3}\,I$$

The singularity

$$z = 2$$

will be skipped because the singularity is not in our region.
Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\, \mathrm{f}(z))) = \frac{4}{3}\,\pi$$

The solution is

$$\int_0^{2\pi} \frac{\cos(x)}{\frac{5}{4} - \cos(x)} \, dx = \frac{4}{3}\,\pi$$

We can try to solve it using real calculus and obtain the result

$$\int_0^{2\,\pi} \frac{\cos(x)}{\frac{5}{4} - \cos(x)}\, dx = \frac{4}{3}\,\pi$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

1.12 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\,dx$$

Hint.

$$\mathit{Type\_II}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z^2+1}{z^4+1}$$

We find singularities

$[\{z = -\frac{1}{2}\sqrt{2} - \frac{1}{2} I\sqrt{2}\}, \{z = \frac{1}{2}\sqrt{2} + \frac{1}{2} I\sqrt{2}\}, \{z = -\frac{1}{2}\sqrt{2} + \frac{1}{2} I\sqrt{2}\}, \{z = \frac{1}{2}\sqrt{2} - \frac{1}{2} I\sqrt{2}\}]$
The singularity

$$z = -\frac{1}{2}\sqrt{2} - \frac{1}{2} I\sqrt{2}$$

will be skipped because the singularity is not in our region.
The singularity

$$z = \frac{1}{2}\sqrt{2} + \frac{1}{2} I\sqrt{2}$$

is in our region and we will add the following residue

$$\mathrm{res}(\frac{1}{2}\sqrt{2} + \frac{1}{2} I\sqrt{2},\, \mathrm{f}(z)) = \frac{1+I}{2\,I\sqrt{2} - 2\sqrt{2}}$$

The singularity

$$z = -\frac{1}{2}\sqrt{2} + \frac{1}{2} I\sqrt{2}$$

is in our region and we will add the following residue

$$\mathrm{res}(-\frac{1}{2}\sqrt{2} + \frac{1}{2} I\sqrt{2},\, \mathrm{f}(z)) = \frac{1-I}{2\,I\sqrt{2} + 2\sqrt{2}}$$

The singularity

$$z = \frac{1}{2}\sqrt{2} - \frac{1}{2} I\sqrt{2}$$

will be skipped because the singularity is not in our region.

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\, \mathrm{f}(z))) = 2\,I\,\pi\,(\frac{1+I}{2\,I\sqrt{2} - 2\sqrt{2}} + \frac{1-I}{2\,I\sqrt{2} + 2\sqrt{2}})$$

The solution is

$$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\,dx = 2\,I\,\pi\,(\frac{1+I}{2\,I\,\sqrt{2}-2\,\sqrt{2}} + \frac{1-I}{2\,I\,\sqrt{2}+2\,\sqrt{2}})$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\,dx = \sqrt{2}\,\pi$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

1.13 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{x^2-1}{(x^2+1)^2}\,dx$$

Hint.

$$Type\_II$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z^2-1}{(z^2+1)^2}$$

We find singularities

$$[\{z=-I\},\,\{z=I\}]$$

The singularity

$$z=-I$$

will be skipped because the singularity is not in our region.
The singularity

$$z=I$$

is in our region and we will add the following residue

$$\mathrm{res}(I,\,\mathrm{f}(z))=0$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\,\mathrm{f}(z)))=0$$

The solution is

$$\int_{-\infty}^{\infty} \frac{x^2-1}{(x^2+1)^2}\,dx=0$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{x^2-1}{(x^2+1)^2}\,dx=0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

1.14 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10\,x^2 + 9}\,dx$$

Hint.

$$\mathit{Type\_II}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z^2 - z + 2}{z^4 + 10\,z^2 + 9}$$

We find singularities

$$[\{z = 3\,I\},\, \{z = -3\,I\},\, \{z = -I\},\, \{z = I\}]$$

The singularity

$$z = 3\,I$$

is in our region and we will add the following residue

$$\mathrm{res}(3\,I,\, \mathrm{f}(z)) = \frac{1}{16} - \frac{7}{48}\,I$$

The singularity

$$z = -3\,I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = -I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = I$$

is in our region and we will add the following residue

$$\mathrm{res}(I,\, \mathrm{f}(z)) = -\frac{1}{16} - \frac{1}{16}\,I$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\, \mathrm{f}(z))) = \frac{5}{12}\,\pi$$

The solution is

$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10\,x^2 + 9}\,dx = \frac{5}{12}\,\pi$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10\,x^2 + 9}\,dx = \frac{5}{12}\,\pi$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

1.15 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)^2}\, dx$$

Hint.

$$\textit{Type\_II}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{1}{(z^2+1)(z^2+4)^2}$$

We find singularities

$$[\{z = 2\,I\}, \{z = -I\}, \{z = I\}, \{z = -2\,I\}]$$

The singularity

$$z = 2\,I$$

is in our region and we will add the following residue

$$\mathrm{res}(2\,I,\, \mathrm{f}(z)) = \frac{11}{288}\, I$$

The singularity

$$z = -I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = I$$

is in our region and we will add the following residue

$$\mathrm{res}(I,\, \mathrm{f}(z)) = -\frac{1}{18}\, I$$

The singularity

$$z = -2\,I$$

will be skipped because the singularity is not in our region.

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\, \mathrm{f}(z))) = \frac{5}{144}\,\pi$$

The solution is

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)^2}\, dx = \frac{5}{144}\,\pi$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)^2}\, dx = \frac{5}{144}\,\pi$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

1.16 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\, dx$$

Hint.

$$\mathit{Type\_II}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{1}{(z^2+1)(z^2+9)}$$

We find singularities

$$[\{z = 3\,I\}, \{z = -3\,I\}, \{z = -I\}, \{z = I\}]$$

The singularity

$$z = 3\,I$$

is in our region and we will add the following residue

$$\mathrm{res}(3\,I, \mathrm{f}(z)) = \frac{1}{48}\,I$$

The singularity

$$z = -3\,I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = -I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = I$$

is in our region and we will add the following residue

$$\mathrm{res}(I, \mathrm{f}(z)) = -\frac{1}{16}\,I$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z, \mathrm{f}(z))) = \frac{1}{12}\,\pi$$

The solution is

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\,dx = \frac{1}{12}\,\pi$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\,dx = \frac{1}{12}\,\pi$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

1.17 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{1}{(x-I)\,(x-2\,I)}\,dx$$

Hint.

$$\textit{Type\_II}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{1}{(z-I)\,(z-2\,I)}$$

We find singularities

$$[\{z = 2\,I\},\, \{z = I\}]$$

The singularity

$$z = 2\,I$$

is in our region and we will add the following residue

$$\mathrm{res}(2\,I,\, \mathrm{f}(z)) = -I$$

The singularity

$$z = I$$

is in our region and we will add the following residue

$$\mathrm{res}(I,\, \mathrm{f}(z)) = I$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\, \mathrm{f}(z))) = 0$$

The solution is

$$\int_{-\infty}^{\infty} \frac{1}{(x-I)\,(x-2\,I)}\,dx = 0$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{1}{(x-I)\,(x-2\,I)}\,dx = 0$$

Info.

$$\textit{not\_given}$$

Comment.

$$\textit{no\_comment}$$

1.18 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{1}{(x - I)(x - 2I)(x + 3I)}\,dx$$

Hint.

$$Type\_II$$

Solution.
We denote

$$f(z) = \frac{1}{(z - I)(z - 2I)(z + 3I)}$$

We find singularities

$$[\{z = 2I\}, \{z = -3I\}, \{z = I\}]$$

The singularity

$$z = 2I$$

is in our region and we will add the following residue

$$\mathrm{res}(2I, f(z)) = \frac{-1}{5}$$

The singularity

$$z = -3I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = I$$

is in our region and we will add the following residue

$$\mathrm{res}(I, f(z)) = \frac{1}{4}$$

Our sum is

$$2I\pi\,(\sum \mathrm{res}(z, f(z))) = \frac{1}{10}I\pi$$

The solution is

$$\int_{-\infty}^{\infty} \frac{1}{(x - I)(x - 2I)(x + 3I)}\,dx = \frac{1}{10}I\pi$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{1}{(x - I)\,(x - 2\,I)\,(x + 3\,I)}\,dx = \frac{1}{10}\,I\,\pi$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

1.19 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

Hint.

$$\mathit{Type\_II}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{1}{z^2+1}$$

We find singularities

$$[\{z=-I\},\,\{z=I\}]$$

The singularity

$$z=-I$$

will be skipped because the singularity is not in our region.
The singularity

$$z=I$$

is in our region and we will add the following residue

$$\mathrm{res}(I,\,\mathrm{f}(z)) = -\frac{1}{2}\,I$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\,\mathrm{f}(z))) = \pi$$

The solution is

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$$

Info.

$$\mathit{not\_given}$$

Comment.

$$\mathit{no\_comment}$$

1.20 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{1}{x^3+1}\,dx$$

Hint.

$$Type\_II$$

Solution.
We denote

$$f(z) = \frac{1}{z^3+1}$$

We find singularities

$$[\{z=-1\},\,\{z=\frac{1}{2}-\frac{1}{2}\,I\,\sqrt{3}\},\,\{z=\frac{1}{2}+\frac{1}{2}\,I\,\sqrt{3}\}]$$

The singularity

$$z=-1$$

is on the real line and we will add one half of the following residue

$$\mathrm{res}(-1,\,f(z)) = \frac{1}{3}$$

The singularity

$$z=\frac{1}{2}-\frac{1}{2}\,I\,\sqrt{3}$$

will be skipped because the singularity is not in our region.
The singularity

$$z=\frac{1}{2}+\frac{1}{2}\,I\,\sqrt{3}$$

is in our region and we will add the following residue

$$\mathrm{res}(\frac{1}{2}+\frac{1}{2}\,I\,\sqrt{3},\,f(z)) = 2\,\frac{1}{3\,I\,\sqrt{3}-3}$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\,f(z))) = 2\,I\,\pi\,(\frac{1}{6}+2\,\frac{1}{3\,I\,\sqrt{3}-3})$$

The solution is

$$\int_{-\infty}^{\infty} \frac{1}{x^3+1}\,dx = 2\,I\,\pi\,(\frac{1}{6}+2\,\frac{1}{3\,I\,\sqrt{3}-3})$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{1}{x^3+1}\,dx = \int_{-\infty}^{\infty} \frac{1}{x^3+1}\,dx$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

1.21 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{1}{x^6+1}\, dx$$

Hint.

$$\mathit{Type\_II}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{1}{z^6+1}$$

We find singularities

$[\{z = -I\}, \{z = I\}, \{z = \frac{1}{2}\sqrt{2+2\,I\,\sqrt{3}}\}, \{z = -\frac{1}{2}\sqrt{2+2\,I\,\sqrt{3}}\}, \{z = \frac{1}{2}\sqrt{2-2\,I\,\sqrt{3}}\},$
$\{z = -\frac{1}{2}\sqrt{2-2\,I\,\sqrt{3}}\}]$

The singularity

$$z = -I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = I$$

is in our region and we will add the following residue

$$\mathrm{res}(I,\, \mathrm{f}(z)) = -\frac{1}{6}\, I$$

The singularity

$$z = \frac{1}{2}\sqrt{2+2\,I\,\sqrt{3}}$$

is in our region and we will add the following residue

$$\mathrm{res}(\frac{1}{2}\sqrt{2+2\,I\,\sqrt{3}},\, \mathrm{f}(z)) = \frac{16}{3}\,\frac{1}{(2+2\,I\,\sqrt{3})^{(5/2)}}$$

The singularity

$$z = -\frac{1}{2}\sqrt{2+2\,I\,\sqrt{3}}$$

will be skipped because the singularity is not in our region.
The singularity

$$z = \frac{1}{2}\sqrt{2-2\,I\,\sqrt{3}}$$

will be skipped because the singularity is not in our region.
The singularity

$$z = -\frac{1}{2}\sqrt{2 - 2\,I\,\sqrt{3}}$$

is in our region and we will add the following residue

$$\mathrm{res}(-\frac{1}{2}\sqrt{2 - 2\,I\,\sqrt{3}},\ \mathrm{f}(z)) = -\frac{16}{3}\,\frac{1}{(2 - 2\,I\,\sqrt{3})^{(5/2)}}$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\ \mathrm{f}(z))) = 2\,I\,\pi\,(-\frac{1}{6}\,I + \frac{16}{3}\,\frac{1}{(2 + 2\,I\,\sqrt{3})^{(5/2)}} - \frac{16}{3}\,\frac{1}{(2 - 2\,I\,\sqrt{3})^{(5/2)}})$$

The solution is

$$\int_{-\infty}^{\infty} \frac{1}{x^6 + 1}\,dx = 2\,I\,\pi\,(-\frac{1}{6}\,I + \frac{16}{3}\,\frac{1}{(2 + 2\,I\,\sqrt{3})^{(5/2)}} - \frac{16}{3}\,\frac{1}{(2 - 2\,I\,\sqrt{3})^{(5/2)}})$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{1}{x^6 + 1}\,dx = \frac{2}{3}\,\pi$$

Info.

not_given

Comment.

no_comment

1.22 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{x}{(x^2 + 4\,x + 13)^2}\,dx$$

Hint.

$$\textit{Type\_II}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z}{(z^2 + 4\,z + 13)^2}$$

We find singularities

$$[\{z = -2 + 3\,I\},\ \{z = -2 - 3\,I\}]$$

The singularity

$$z = -2 + 3\,I$$

is in our region and we will add the following residue

$$\mathrm{res}(-2 + 3\,I,\ \mathrm{f}(z)) = \frac{1}{54}\,I$$

The singularity

$$z = -2 - 3\,I$$

will be skipped because the singularity is not in our region.

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\ \mathrm{f}(z))) = -\frac{1}{27}\,\pi$$

The solution is

$$\int_{-\infty}^{\infty} \frac{x}{(x^2 + 4\,x + 13)^2}\,dx = -\frac{1}{27}\,\pi$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{x}{(x^2 + 4\,x + 13)^2}\,dx = -\frac{1}{27}\,\pi$$

Info.

$$\textit{not\_given}$$

Comment.

$$\textit{no\_comment}$$

1.23 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+9)}\, dx$$

Hint.

$$\textit{Type\_II}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z^2}{(z^2+1)(z^2+9)}$$

We find singularities

$$[\{z = 3\,I\}, \{z = -3\,I\}, \{z = -I\}, \{z = I\}]$$

The singularity

$$z = 3\,I$$

is in our region and we will add the following residue

$$\mathrm{res}(3\,I, \mathrm{f}(z)) = -\frac{3}{16}\,I$$

The singularity

$$z = -3\,I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = -I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = I$$

is in our region and we will add the following residue

$$\mathrm{res}(I, \mathrm{f}(z)) = \frac{1}{16}\,I$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z, \mathrm{f}(z))) = \frac{1}{4}\,\pi$$

The solution is

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+9)}\, dx = \frac{1}{4}\,\pi$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+9)}\, dx = \frac{1}{4}\,\pi$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

1.24 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^3}\, dx$$

Hint.

$$\mathit{Type\_II}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z^2}{(z^2+1)^3}$$

We find singularities

$$[\{z = -I\},\, \{z = I\}]$$

The singularity

$$z = -I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = I$$

is in our region and we will add the following residue

$$\mathrm{res}(I,\, \mathrm{f}(z)) = -\frac{1}{16}\, I$$

Our sum is

$$2\, I\, \pi \left(\sum \mathrm{res}(z,\, \mathrm{f}(z))\right) = \frac{1}{8}\, \pi$$

The solution is

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^3}\, dx = \frac{1}{8}\, \pi$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^3}\, dx = \frac{1}{8}\, \pi$$

Info.

$$\mathit{not\_given}$$

Comment.

$$\mathit{no\_comment}$$

1.25 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2}\, dx$$

Hint.

$$\mathit{Type\_II}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z^2}{(z^2+4)^2}$$

We find singularities

$$[\{z = 2\,I\},\, \{z = -2\,I\}]$$

The singularity

$$z = 2\,I$$

is in our region and we will add the following residue

$$\mathrm{res}(2\,I,\, \mathrm{f}(z)) = -\frac{1}{8}\,I$$

The singularity

$$z = -2\,I$$

will be skipped because the singularity is not in our region.

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\, \mathrm{f}(z))) = \frac{1}{4}\,\pi$$

The solution is

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2}\, dx = \frac{1}{4}\,\pi$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2}\, dx = \frac{1}{4}\,\pi$$

Info.

$$\mathit{not\_given}$$

Comment.

$$\mathit{no\_comment}$$

1.26 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{(x^3 + 5\,x)\, e^{(I\,x)}}{x^4 + 10\,x^2 + 9}\, dx$$

Hint.

$$\mathit{Type\_III}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{(z^3 + 5\,z)\, e^{(I\,z)}}{z^4 + 10\,z^2 + 9}$$

We find singularities

$$[\{z = 3\,I\},\, \{z = -3\,I\},\, \{z = -I\},\, \{z = I\}]$$

The singularity

$$z = 3\,I$$

is in our region and we will add the following residue

$$\mathrm{res}(3\,I,\, \mathrm{f}(z)) = \frac{1}{4}\, e^{(-3)}$$

The singularity

$$z = -3\,I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = -I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = I$$

is in our region and we will add the following residue

$$\mathrm{res}(I,\, \mathrm{f}(z)) = \frac{1}{4}\, e^{(-1)}$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\, \mathrm{f}(z))) = 2\,I\,\pi\,(\frac{1}{4}\, e^{(-3)} + \frac{1}{4}\, e^{(-1)})$$

The solution is

$$\int_{-\infty}^{\infty} \frac{(x^3 + 5\,x)\,e^{(I\,x)}}{x^4 + 10\,x^2 + 9}\,dx = 2\,I\,\pi\,(\frac{1}{4}\,e^{(-3)} + \frac{1}{4}\,e^{(-1)})$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{(x^3 + 5\,x)\,e^{(I\,x)}}{x^4 + 10\,x^2 + 9}\,dx = \frac{1}{2}\,I\,\pi\cosh(1)+\frac{1}{2}\,I\,\pi\cosh(3)-\frac{1}{2}\,I\,\pi\sinh(3)-\frac{1}{2}\,I\,\pi\sinh(1)$$

Info.

$$\mathit{not\_given}$$

Comment.

$$\mathit{no\_comment}$$

1.27 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}\,(x^2-1)}{x\,(x^2+1)}\,dx$$

Hint.

$$Type\_III$$

Solution.
We denote

$$f(z) = \frac{e^{(I\,z)}\,(z^2-1)}{z\,(z^2+1)}$$

We find singularities

$$[\{z=0\},\,\{z=-I\},\,\{z=I\}]$$

The singularity

$$z = 0$$

is on the real line and we will add one half of the following residue

$$\mathrm{res}(0,\,f(z)) = -1$$

The singularity

$$z = -I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = I$$

is in our region and we will add the following residue

$$\mathrm{res}(I,\,f(z)) = e^{(-1)}$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\,f(z))) = 2\,I\,\pi\,(-\frac{1}{2} + e^{(-1)})$$

The solution is

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}\,(x^2-1)}{x\,(x^2+1)}\,dx = 2\,I\,\pi\,(-\frac{1}{2} + e^{(-1)})$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}\,(x^2-1)}{x\,(x^2+1)}\,dx = \int_{-\infty}^{\infty} \frac{(\cos(x) + I\sin(x))\,(x^2-1)}{x\,(x^2+1)}\,dx$$

Info.

not_given

Comment.

no_comment

1.28 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{(x^2+4)\,(x-1)}\,dx$$

Hint.

$$\mathit{Type\_III}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{(z^2+4)\,(z-1)}$$

We find singularities

$$[\{z = 2\,I\},\,\{z = -2\,I\},\,\{z = 1\}]$$

The singularity

$$z = 2\,I$$

is in our region and we will add the following residue

$$\mathrm{res}(2\,I,\,\mathrm{f}(z)) = (-\frac{1}{10} + \frac{1}{20}\,I)\,e^{(-2)}$$

The singularity

$$z = -2\,I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = 1$$

is on the real line and we will add one half of the following residue

$$\mathrm{res}(1,\,\mathrm{f}(z)) = \frac{1}{5}\,e^{I}$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\,\mathrm{f}(z))) = 2\,I\,\pi\,((-\frac{1}{10} + \frac{1}{20}\,I)\,e^{(-2)} + \frac{1}{10}\,e^{I})$$

The solution is

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{(x^2+4)\,(x-1)}\,dx = 2\,I\,\pi\,((-\frac{1}{10} + \frac{1}{20}\,I)\,e^{(-2)} + \frac{1}{10}\,e^{I})$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{(x^2+4)\,(x-1)}\,dx = \int_{-\infty}^{\infty} \frac{\cos(x) + I\sin(x)}{(x^2+4)\,(x-1)}\,dx$$

Info.

not_given

Comment.

no_comment

1.29 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x\,(x^2+1)}\,dx$$

Hint.

Type_III

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{z\,(z^2+1)}$$

We find singularities

$$[\{z=0\},\,\{z=-I\},\,\{z=I\}]$$

The singularity

$$z=0$$

is on the real line and we will add one half of the following residue

$$\mathrm{res}(0,\,\mathrm{f}(z)) = 1$$

The singularity

$$z=-I$$

will be skipped because the singularity is not in our region.
The singularity

$$z=I$$

is in our region and we will add the following residue

$$\mathrm{res}(I,\,\mathrm{f}(z)) = -\frac{1}{2}\,e^{(-1)}$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\,\mathrm{f}(z))) = 2\,I\,\pi\,(\frac{1}{2}-\frac{1}{2}\,e^{(-1)})$$

The solution is

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x\,(x^2+1)}\,dx = 2\,I\,\pi\,(\frac{1}{2}-\frac{1}{2}\,e^{(-1)})$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x\,(x^2+1)}\,dx = \int_{-\infty}^{\infty} \frac{\cos(x) + I\sin(x)}{x\,(x^2+1)}\,dx$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

1.30 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x\,(x^2+9)^2}\,dx$$

Hint.

$$\mathit{Type\_III}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{z\,(z^2+9)^2}$$

We find singularities

$$[\{z=0\},\,\{z=3\,I\},\,\{z=-3\,I\}]$$

The singularity

$$z = 0$$

is on the real line and we will add one half of the following residue

$$\mathrm{res}(0,\,\mathrm{f}(z)) = \frac{1}{81}$$

The singularity

$$z = 3\,I$$

is in our region and we will add the following residue

$$\mathrm{res}(3\,I,\,\mathrm{f}(z)) = -\frac{5}{324}\,e^{(-3)}$$

The singularity

$$z = -3\,I$$

will be skipped because the singularity is not in our region.

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\,\mathrm{f}(z))) = 2\,I\,\pi\,(\frac{1}{162} - \frac{5}{324}\,e^{(-3)})$$

The solution is

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x\,(x^2+9)^2}\,dx = 2\,I\,\pi\,(\frac{1}{162} - \frac{5}{324}\,e^{(-3)})$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x\,(x^2+9)^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos(x) + I\sin(x)}{x\,(x^2+9)^2}\,dx$$

Info.

not_given

Comment.

no_comment

1.31 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x\,(x^2-2\,x+2)}\,dx$$

Hint.

$$\mathit{Type\_III}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{z\,(z^2-2\,z+2)}$$

We find singularities

$$[\{z=0\},\,\{z=1+I\},\,\{z=1-I\}]$$

The singularity

$$z=0$$

is on the real line and we will add one half of the following residue

$$\mathrm{res}(0,\,\mathrm{f}(z)) = \frac{1}{2}$$

The singularity

$$z=1+I$$

is in our region and we will add the following residue

$$\mathrm{res}(1+I,\,\mathrm{f}(z)) = (-\frac{1}{4}-\frac{1}{4}\,I)\,e^I\,e^{(-1)}$$

The singularity

$$z=1-I$$

will be skipped because the singularity is not in our region.

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\,\mathrm{f}(z))) = 2\,I\,\pi\,(\frac{1}{4}+(-\frac{1}{4}-\frac{1}{4}\,I)\,e^I\,e^{(-1)})$$

The solution is

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x\,(x^2-2\,x+2)}\,dx = 2\,I\,\pi\,(\frac{1}{4}+(-\frac{1}{4}-\frac{1}{4}\,I)\,e^I\,e^{(-1)})$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x\,(x^2 - 2\,x + 2)}\,dx = \int_{-\infty}^{\infty} \frac{\cos(x) + I\sin(x)}{x\,(x^2 - 2\,x + 2)}\,dx$$

Info.

not_given

Comment.

no_comment

1.32 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^2+1}\,dx$$

Hint.

$$\mathit{Type\_III}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{z^2+1}$$

We find singularities

$$[\{z=-I\},\,\{z=I\}]$$

The singularity

$$z=-I$$

will be skipped because the singularity is not in our region.
The singularity

$$z=I$$

is in our region and we will add the following residue

$$\mathrm{res}(I,\,\mathrm{f}(z)) = -\frac{1}{2}\,I\,e^{(-1)}$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\,\mathrm{f}(z))) = \pi\,e^{(-1)}$$

The solution is

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^2+1}\,dx = \pi\,e^{(-1)}$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^2+1}\,dx = -\pi\sinh(1)+\pi\cosh(1)$$

Info.

$$\mathit{not\_given}$$

Comment.

$$\mathit{no\_comment}$$

1.33 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^2 + 4\,x + 20}\,dx$$

Hint.

$$\textit{Type\_III}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{z^2 + 4\,z + 20}$$

We find singularities

$$[\{z = -2 - 4\,I\}, \{z = -2 + 4\,I\}]$$

The singularity

$$z = -2 - 4\,I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = -2 + 4\,I$$

is in our region and we will add the following residue

$$\mathrm{res}(-2 + 4\,I, \mathrm{f}(z)) = -\frac{1}{8}\,\frac{I\,e^{(-4)}}{(e^I)^2}$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z, \mathrm{f}(z))) = \frac{1}{4}\,\frac{\pi\,e^{(-4)}}{(e^I)^2}$$

The solution is

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^2 + 4\,x + 20}\,dx = \frac{1}{4}\,\frac{\pi\,e^{(-4)}}{(e^I)^2}$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^2 + 4\,x + 20}\,dx = -\frac{1}{4}\,I\,\pi\sin(2 - 4\,I) + \frac{1}{4}\,\pi\cos(2 - 4\,I)$$

Info.

$$\textit{not\_given}$$

Comment.

$$\textit{no\_comment}$$

1.34 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^2 - 5\,x + 6}\,dx$$

Hint.

$$\mathit{Type\_III}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{z^2 - 5\,z + 6}$$

We find singularities

$$[\{z = 2\},\,\{z = 3\}]$$

The singularity

$$z = 2$$

is on the real line and we will add one half of the following residue

$$\mathrm{res}(2,\,\mathrm{f}(z)) = -(e^I)^2$$

The singularity

$$z = 3$$

is on the real line and we will add one half of the following residue

$$\mathrm{res}(3,\,\mathrm{f}(z)) = (e^I)^3$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\,\mathrm{f}(z))) = 2\,I\,\pi\,(-\frac{1}{2}\,(e^I)^2 + \frac{1}{2}\,(e^I)^3)$$

The solution is

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^2 - 5\,x + 6}\,dx = 2\,I\,\pi\,(-\frac{1}{2}\,(e^I)^2 + \frac{1}{2}\,(e^I)^3)$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^2 - 5\,x + 6}\,dx = \int_{-\infty}^{\infty} \frac{\cos(x) + I\sin(x)}{x^2 - 5\,x + 6}\,dx$$

Info.

$$\mathit{not\_given}$$

Comment.

$$\mathit{no\_comment}$$

1.35 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^3+1}\,dx$$

Hint.

$$\mathit{Type\_III}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{z^3+1}$$

We find singularities

$$[\{z=-1\},\ \{z=\frac{1}{2}-\frac{1}{2}\,I\,\sqrt{3}\},\ \{z=\frac{1}{2}+\frac{1}{2}\,I\,\sqrt{3}\}]$$

The singularity

$$z=-1$$

is on the real line and we will add one half of the following residue

$$\mathrm{res}(-1,\ \mathrm{f}(z)) = \frac{1}{3}\,\frac{1}{e^I}$$

The singularity

$$z=\frac{1}{2}-\frac{1}{2}\,I\,\sqrt{3}$$

will be skipped because the singularity is not in our region.
The singularity

$$z=\frac{1}{2}+\frac{1}{2}\,I\,\sqrt{3}$$

is in our region and we will add the following residue

$$\mathrm{res}(\frac{1}{2}+\frac{1}{2}\,I\,\sqrt{3},\ \mathrm{f}(z)) = 2\,\frac{(-1)^{(1/2\,\frac{1}{\pi})}}{3\,I\,\sqrt{e^{(\sqrt{3})}}\,\sqrt{3}-3\,\sqrt{e^{(\sqrt{3})}}}$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\ \mathrm{f}(z))) = 2\,I\,\pi\,\left(\frac{1}{6}\,\frac{1}{e^I}+2\,\frac{(-1)^{(1/2\,\frac{1}{\pi})}}{3\,I\,\sqrt{e^{(\sqrt{3})}}\,\sqrt{3}-3\,\sqrt{e^{(\sqrt{3})}}}\right)$$

The solution is

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^3+1}\,dx = 2\,I\,\pi\,\left(\frac{1}{6}\,\frac{1}{e^I}+2\,\frac{(-1)^{(1/2\,\frac{1}{\pi})}}{3\,I\,\sqrt{e^{(\sqrt{3})}}\,\sqrt{3}-3\,\sqrt{e^{(\sqrt{3})}}}\right)$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^3+1}\,dx = \int_{-\infty}^{\infty} \frac{\cos(x) + I\sin(x)}{x^3+1}\,dx$$

Info.

not_given

Comment.

no_comment

1.36 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^4 - 1}\,dx$$

Hint.

$$Type\_III$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{z^4 - 1}$$

We find singularities

$$[\{z = -1\}, \{z = -I\}, \{z = I\}, \{z = 1\}]$$

The singularity

$$z = -1$$

is on the real line and we will add one half of the following residue

$$\mathrm{res}(-1, \mathrm{f}(z)) = -\frac{1}{4}\,\frac{1}{e^I}$$

The singularity

$$z = -I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = I$$

is in our region and we will add the following residue

$$\mathrm{res}(I, \mathrm{f}(z)) = \frac{1}{4}\,I\,e^{(-1)}$$

The singularity

$$z = 1$$

is on the real line and we will add one half of the following residue

$$\mathrm{res}(1, \mathrm{f}(z)) = \frac{1}{4}\,e^I$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z, \mathrm{f}(z))) = 2\,I\,\pi\,(-\frac{1}{8}\,\frac{1}{e^I} + \frac{1}{4}\,I\,e^{(-1)} + \frac{1}{8}\,e^I)$$

The solution is

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^4 - 1}\,dx = 2\,I\,\pi\,(-\frac{1}{8}\,\frac{1}{e^I} + \frac{1}{4}\,I\,e^{(-1)} + \frac{1}{8}\,e^I)$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^4 - 1}\,dx = \int_{-\infty}^{\infty} \frac{\cos(x) + I\sin(x)}{x^4 - 1}\,dx$$

Info.

not_given

Comment.

no_comment

1.37 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x}\,dx$$

Hint.

$$Type\_III$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{z}$$

We find singularities

$$[\{z = 0\}]$$

The singularity

$$z = 0$$

is on the real line and we will add one half of the following residue

$$\mathrm{res}(0,\, \mathrm{f}(z)) = 1$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\, \mathrm{f}(z))) = I\,\pi$$

The solution is

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x}\,dx = I\,\pi$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x}\,dx = \int_{-\infty}^{\infty} \frac{\cos(x) + I\sin(x)}{x}\,dx$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

1.38 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{x\, e^{(I\, x)}}{1 - x^4}\, dx$$

Hint.

$$Type\_III$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z\, e^{(I\, z)}}{1 - z^4}$$

We find singularities

$$[\{z = -1\}, \{z = -I\}, \{z = I\}, \{z = 1\}]$$

The singularity

$$z = -1$$

is on the real line and we will add one half of the following residue

$$\mathrm{res}(-1,\, \mathrm{f}(z)) = -\frac{1}{4}\, \frac{1}{e^I}$$

The singularity

$$z = -I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = I$$

is in our region and we will add the following residue

$$\mathrm{res}(I,\, \mathrm{f}(z)) = \frac{1}{4}\, e^{(-1)}$$

The singularity

$$z = 1$$

is on the real line and we will add one half of the following residue

$$\mathrm{res}(1,\, \mathrm{f}(z)) = -\frac{1}{4}\, e^I$$

Our sum is

$$2\, I\, \pi\, (\sum \mathrm{res}(z,\, \mathrm{f}(z))) = 2\, I\, \pi\, (-\frac{1}{8}\, \frac{1}{e^I} + \frac{1}{4}\, e^{(-1)} - \frac{1}{8}\, e^I)$$

The solution is

$$\int_{-\infty}^{\infty} \frac{x\, e^{(I\, x)}}{1 - x^4}\, dx = 2\, I\, \pi\, (-\frac{1}{8}\, \frac{1}{e^I} + \frac{1}{4}\, e^{(-1)} - \frac{1}{8}\, e^I)$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{x\, e^{(I\, x)}}{1 - x^4}\, dx = \int_{-\infty}^{\infty} -\frac{x\, (\cos(x) + I\, \sin(x))}{-1 + x^4}\, dx$$

Info.

not_given

Comment.

no_comment

1.39 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{x\, e^{(I\,x)}}{x^2 + 4\,x + 20}\, dx$$

Hint.

$$\mathit{Type\_III}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z\, e^{(I\,z)}}{z^2 + 4\,z + 20}$$

We find singularities

$$[\{z = -2 - 4\,I\},\, \{z = -2 + 4\,I\}]$$

The singularity

$$z = -2 - 4\,I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = -2 + 4\,I$$

is in our region and we will add the following residue

$$\mathrm{res}(-2 + 4\,I,\, \mathrm{f}(z)) = -\frac{1}{8}\,\frac{I\,(4\,I\,e^{(-4)} - 2\,e^{(-4)})}{(e^I)^2}$$

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\, \mathrm{f}(z))) = \frac{1}{4}\,\frac{\pi\,(4\,I\,e^{(-4)} - 2\,e^{(-4)})}{(e^I)^2}$$

The solution is

$$\int_{-\infty}^{\infty} \frac{x\, e^{(I\,x)}}{x^2 + 4\,x + 20}\, dx = \frac{1}{4}\,\frac{\pi\,(4\,I\,e^{(-4)} - 2\,e^{(-4)})}{(e^I)^2}$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{x\, e^{(I\,x)}}{x^2 + 4\,x + 20}\, dx =$$
$$\pi \sin(2 - 4\,I) + I\,\pi \cos(2 - 4\,I) + \frac{1}{2}\,I\,\pi \sin(2 - 4\,I) - \frac{1}{2}\,\pi \cos(2 - 4\,I)$$

Info.

not_given

Comment.

no_comment

1.40 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{x\, e^{(I\, x)}}{x^2+9}\, dx$$

Hint.

$$\textit{Type\_III}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z\, e^{(I\, z)}}{z^2+9}$$

We find singularities

$$[\{z = 3\, I\},\, \{z = -3\, I\}]$$

The singularity

$$z = 3\, I$$

is in our region and we will add the following residue

$$\mathrm{res}(3\, I,\, \mathrm{f}(z)) = \frac{1}{2}\, e^{(-3)}$$

The singularity

$$z = -3\, I$$

will be skipped because the singularity is not in our region.

Our sum is

$$2\, I\, \pi\, (\sum \mathrm{res}(z,\, \mathrm{f}(z))) = I\, \pi\, e^{(-3)}$$

The solution is

$$\int_{-\infty}^{\infty} \frac{x\, e^{(I\, x)}}{x^2+9}\, dx = I\, \pi\, e^{(-3)}$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{x\, e^{(I\, x)}}{x^2+9}\, dx = I\, \pi \cosh(3) - I\, \pi \sinh(3)$$

Info.

$$\textit{not\_given}$$

Comment.

$$\textit{no\_comment}$$

1.41 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{x\, e^{(I\, x)}}{x^2 - 2\, x + 10}\, dx$$

Hint.

$$\mathit{Type\_III}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z\, e^{(I\, z)}}{z^2 - 2\, z + 10}$$

We find singularities

$$[\{z = 1 - 3\, I\},\ \{z = 1 + 3\, I\}]$$

The singularity

$$z = 1 - 3\, I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = 1 + 3\, I$$

is in our region and we will add the following residue

$$\mathrm{res}(1 + 3\, I,\ \mathrm{f}(z)) = -\frac{1}{6}\, I\, (3\, I\, e^I\, e^{(-3)} + e^I\, e^{(-3)})$$

Our sum is

$$2\, I\, \pi\, (\sum \mathrm{res}(z,\ \mathrm{f}(z))) = \frac{1}{3}\, \pi\, (3\, I\, e^I\, e^{(-3)} + e^I\, e^{(-3)})$$

The solution is

$$\int_{-\infty}^{\infty} \frac{x\, e^{(I\, x)}}{x^2 - 2\, x + 10}\, dx = \frac{1}{3}\, \pi\, (3\, I\, e^I\, e^{(-3)} + e^I\, e^{(-3)})$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{x\, e^{(I\, x)}}{x^2 - 2\, x + 10}\, dx =$$
$$-\pi \sin(1 + 3\, I) + I\, \pi \cos(1 + 3\, I) + \frac{1}{3}\, I\, \pi \sin(1 + 3\, I) + \frac{1}{3}\, \pi \cos(1 + 3\, I)$$

Info.

$$\mathit{not\_given}$$

Comment.

$no\_comment$

1.42 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{x\, e^{(I\, x)}}{x^2 - 5\, x + 6}\, dx$$

Hint.

$$Type\_III$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z\, e^{(I\, z)}}{z^2 - 5\, z + 6}$$

We find singularities

$$[\{z = 2\},\, \{z = 3\}]$$

The singularity

$$z = 2$$

is on the real line and we will add one half of the following residue

$$\mathrm{res}(2,\, \mathrm{f}(z)) = -2\, (e^I)^2$$

The singularity

$$z = 3$$

is on the real line and we will add one half of the following residue

$$\mathrm{res}(3,\, \mathrm{f}(z)) = 3\, (e^I)^3$$

Our sum is

$$2\, I\, \pi\, (\sum \mathrm{res}(z,\, \mathrm{f}(z))) = 2\, I\, \pi\, (-(e^I)^2 + \frac{3}{2}\, (e^I)^3)$$

The solution is

$$\int_{-\infty}^{\infty} \frac{x\, e^{(I\, x)}}{x^2 - 5\, x + 6}\, dx = 2\, I\, \pi\, (-(e^I)^2 + \frac{3}{2}\, (e^I)^3)$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{x\, e^{(I\, x)}}{x^2 - 5\, x + 6}\, dx = \int_{-\infty}^{\infty} \frac{x\, (\cos(x) + I\, \sin(x))}{x^2 - 5\, x + 6}\, dx$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

1.43 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^2}\,dx$$

Hint.

$$Type\_III$$

Solution.
We denote

$$f(z) = \frac{e^{(I\,z)}}{z^2}$$

We find singularities

$$[\{z = 0\}]$$

The singularity

$$z = 0$$

is on the real line and is not a simple pole, we cannot count the integral with the residue theorem ...

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\, f(z))) = 2\,I\,\pi\,\infty$$

The solution is

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^2}\,dx = 2\,I\,\pi\,\infty$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^2}\,dx = \infty$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

1.44 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{x^3\, e^{(I\,x)}}{x^4 + 5\,x^2 + 4}\, dx$$

Hint.

$$\mathit{Type\_III}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z^3\, e^{(I\,z)}}{z^4 + 5\,z^2 + 4}$$

We find singularities

$$[\{z = 2\,I\},\ \{z = -I\},\ \{z = I\},\ \{z = -2\,I\}]$$

The singularity

$$z = 2\,I$$

is in our region and we will add the following residue

$$\mathrm{res}(2\,I,\ \mathrm{f}(z)) = \frac{2}{3}\, e^{(-2)}$$

The singularity

$$z = -I$$

will be skipped because the singularity is not in our region.
The singularity

$$z = I$$

is in our region and we will add the following residue

$$\mathrm{res}(I,\ \mathrm{f}(z)) = -\frac{1}{6}\, e^{(-1)}$$

The singularity

$$z = -2\,I$$

will be skipped because the singularity is not in our region.

Our sum is

$$2\,I\,\pi\,(\sum \mathrm{res}(z,\ \mathrm{f}(z))) = 2\,I\,\pi\,(\frac{2}{3}\, e^{(-2)} - \frac{1}{6}\, e^{(-1)})$$

The solution is

$$\int_{-\infty}^{\infty} \frac{x^3\, e^{(I\, x)}}{x^4 + 5\, x^2 + 4}\, dx = 2\, I\, \pi\, (\frac{2}{3}\, e^{(-2)} - \frac{1}{6}\, e^{(-1)})$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{x^3\, e^{(I\, x)}}{x^4 + 5\, x^2 + 4}\, dx = -\frac{1}{3}\, I\, \pi \cosh(1) + \frac{4}{3}\, I\, \pi \cosh(2) - \frac{4}{3}\, I\, \pi \sinh(2) + \frac{1}{3}\, I\, \pi \sinh(1)$$

Info.

not_given

Comment.

no_comment

2 Zero Sum theorem for residues problems

We recall the definition:

Notation. For a function f holomorphic on some neighbourhood of infinity we define

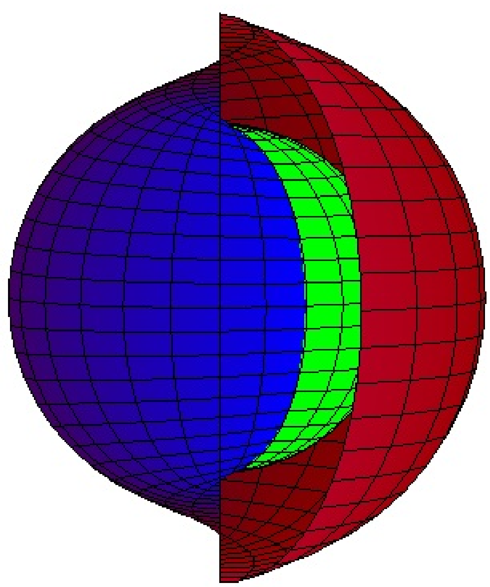
$$\operatorname{res}(f,\infty) = \operatorname{res}\left(\frac{-1}{z^2}\cdot f\left(\frac{1}{z}\right),0\right).$$

We will solve several problems using the following theorem:

Theorem. (Zero Sum theorem for residues) For a function f holomorphic in the extended complex plane $\mathbb{C}\cup\{\infty\}$ with at most finitely many exceptions the sum of residues is zero, i.e.

$$\sum_{w\in\mathbb{C}\cup\{\infty\}} \operatorname{res}(f,w) = 0\,.$$

Notation.(Example $f(z) = 1/z$) Projecting on the Riemann sphere we observe the north pole on globe (i.e. ∞) with the residue -1 and at the south pole (the origin) with residue 1. Green is the zero level on the Riemann sphere while the blue gradually turning red is the imaginary part of $\log z$ demonstrating that Zero Sum theorem holds for $1/z$.



2.1 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z^2+1}{z^4+1}$$

Hint.

$$\mathit{no\_hint}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z^2+1}{z^4+1}$$

We find singularities

$[\{z = \frac{1}{2}\sqrt{2}+\frac{1}{2}I\sqrt{2}\}, \{z = -\frac{1}{2}\sqrt{2}-\frac{1}{2}I\sqrt{2}\}, \{z = \frac{1}{2}\sqrt{2}-\frac{1}{2}I\sqrt{2}\}, \{z = -\frac{1}{2}\sqrt{2}+\frac{1}{2}I\sqrt{2}\}]$

The singularity

$$z = \frac{1}{2}\sqrt{2} + \frac{1}{2}I\sqrt{2}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ \frac{1}{2}\sqrt{2} + \frac{1}{2}I\sqrt{2}) = \frac{1+I}{2I\sqrt{2} - 2\sqrt{2}}$$

The singularity

$$z = -\frac{1}{2}\sqrt{2} - \frac{1}{2}I\sqrt{2}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ -\frac{1}{2}\sqrt{2} - \frac{1}{2}I\sqrt{2}) = \frac{-1-I}{2I\sqrt{2} - 2\sqrt{2}}$$

The singularity

$$z = \frac{1}{2}\sqrt{2} - \frac{1}{2}I\sqrt{2}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ \frac{1}{2}\sqrt{2} - \frac{1}{2}I\sqrt{2}) = \frac{-1+I}{2I\sqrt{2} + 2\sqrt{2}}$$

The singularity

$$z = -\frac{1}{2}\sqrt{2} + \frac{1}{2}I\sqrt{2}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ -\frac{1}{2}\sqrt{2}+\frac{1}{2}\,I\,\sqrt{2}) = \frac{1-I}{2\,I\,\sqrt{2}+2\,\sqrt{2}}$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\ \infty) = \frac{1+I}{2\,I\,\sqrt{2}-2\,\sqrt{2}} + \frac{-1-I}{2\,I\,\sqrt{2}-2\,\sqrt{2}} + \frac{-1+I}{2\,I\,\sqrt{2}+2\,\sqrt{2}} + \frac{1-I}{2\,I\,\sqrt{2}+2\,\sqrt{2}}$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\ z) = 0$$

Info.

not_given

Comment.

no_comment

2.2 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z^2 - 1}{(z^2 + 1)^2}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z^2 - 1}{(z^2 + 1)^2}$$

We find singularities

$$[\{z = -I\}, \{z = I\}]$$

The singularity

$$z = -I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), -I) = 0$$

The singularity

$$z = I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), I) = 0$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z), \infty) = 0$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z), z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.3 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z^2 - z + 2}{z^4 + 10\,z^2 + 9}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$f(z) = \frac{z^2 - z + 2}{z^4 + 10\,z^2 + 9}$$

We find singularities

$$[\{z = -3\,I\},\ \{z = -I\},\ \{z = I\},\ \{z = 3\,I\}]$$

The singularity

$$z = -3\,I$$

adds the following residue

$$\mathrm{res}(f(z),\ -3\,I) = \frac{1}{16} + \frac{7}{48}\,I$$

The singularity

$$z = -I$$

adds the following residue

$$\mathrm{res}(f(z),\ -I) = -\frac{1}{16} + \frac{1}{16}\,I$$

The singularity

$$z = I$$

adds the following residue

$$\mathrm{res}(f(z),\ I) = -\frac{1}{16} - \frac{1}{16}\,I$$

The singularity

$$z = 3\,I$$

adds the following residue

$$\mathrm{res}(f(z),\ 3\,I) = \frac{1}{16} - \frac{7}{48}\,I$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\, \infty) = 0$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\, z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.4 Problem.

Check the Zero Sum theorem for the following function

$$\frac{1}{(z^2+1)(z^2+4)^2}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$f(z) = \frac{1}{(z^2+1)(z^2+4)^2}$$

We find singularities

$$[\{z = -I\}, \{z = 2I\}, \{z = -2I\}, \{z = I\}]$$

The singularity

$$z = -I$$

adds the following residue

$$\mathrm{res}(f(z), -I) = \frac{1}{18} I$$

The singularity

$$z = 2I$$

adds the following residue

$$\mathrm{res}(f(z), 2I) = \frac{11}{288} I$$

The singularity

$$z = -2I$$

adds the following residue

$$\mathrm{res}(f(z), -2I) = -\frac{11}{288} I$$

The singularity

$$z = I$$

adds the following residue

$$\mathrm{res}(f(z), I) = -\frac{1}{18} I$$

At infinity we get the residue

$$\operatorname{res}(\mathrm{f}(z),\, \infty) = 0$$

and finally we obtain the sum

$$\sum \operatorname{res}(\mathrm{f}(z),\, z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.5 Problem.

Check the Zero Sum theorem for the following function

$$\frac{1}{(z^2+1)(z^2+9)}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$f(z) = \frac{1}{(z^2+1)(z^2+9)}$$

We find singularities

$$[\{z = -3\,I\}, \{z = -I\}, \{z = I\}, \{z = 3\,I\}]$$

The singularity

$$z = -3\,I$$

adds the following residue

$$\mathrm{res}(f(z), -3\,I) = -\frac{1}{48}\,I$$

The singularity

$$z = -I$$

adds the following residue

$$\mathrm{res}(f(z), -I) = \frac{1}{16}\,I$$

The singularity

$$z = I$$

adds the following residue

$$\mathrm{res}(f(z), I) = -\frac{1}{16}\,I$$

The singularity

$$z = 3\,I$$

adds the following residue

$$\mathrm{res}(f(z), 3\,I) = \frac{1}{48}\,I$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\, \infty) = 0$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\, z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.6 Problem.

Check the Zero Sum theorem for the following function

$$\frac{1}{(z - I)\,(z - 2\,I)}$$

Hint.

$$\textit{no\_hint}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{1}{(z - I)\,(z - 2\,I)}$$

We find singularities

$$[\{z = 2\,I\},\ \{z = I\}]$$

The singularity

$$z = 2\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ 2\,I) = -I$$

The singularity

$$z = I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ I) = I$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\ \infty) = 0$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\ z) = 0$$

Info.

$$\textit{not\_given}$$

Comment.

$$\textit{no\_comment}$$

2.7 Problem.

Check the Zero Sum theorem for the following function

$$\frac{1}{(z-I)\,(z-2\,I)\,(z+3\,I)}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z)=\frac{1}{(z-I)\,(z-2\,I)\,(z+3\,I)}$$

We find singularities

$$[\{z=-3\,I\},\,\{z=2\,I\},\,\{z=I\}]$$

The singularity

$$z=-3\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,-3\,I)=\frac{-1}{20}$$

The singularity

$$z=2\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,2\,I)=\frac{-1}{5}$$

The singularity

$$z=I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,I)=\frac{1}{4}$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\,\infty)=0$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\,z)=0$$

Info.

not_given

Comment.

$no\_comment$

2.8 Problem.

Check the Zero Sum theorem for the following function

$$\frac{1}{z^2+1}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{1}{z^2+1}$$

We find singularities

$$[\{z = -I\},\, \{z = I\}]$$

The singularity

$$z = -I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, -I) = \frac{1}{2}\, I$$

The singularity

$$z = I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, I) = -\frac{1}{2}\, I$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\, \infty) = 0$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\, z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.9 Problem.

Check the Zero Sum theorem for the following function

$$\frac{1}{z^3+1}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{1}{z^3+1}$$

We find singularities

$$[\{z = -1\}, \{z = \frac{1}{2} - \frac{1}{2} I \sqrt{3}\}, \{z = \frac{1}{2} + \frac{1}{2} I \sqrt{3}\}]$$

The singularity

$$z = -1$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), -1) = \frac{1}{3}$$

The singularity

$$z = \frac{1}{2} - \frac{1}{2} I \sqrt{3}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), \frac{1}{2} - \frac{1}{2} I \sqrt{3}) = -2 \frac{1}{3 I \sqrt{3} + 3}$$

The singularity

$$z = \frac{1}{2} + \frac{1}{2} I \sqrt{3}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), \frac{1}{2} + \frac{1}{2} I \sqrt{3}) = 2 \frac{1}{3 I \sqrt{3} - 3}$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z), \infty) = -\frac{1}{3} + 2 \frac{1}{3 I \sqrt{3} + 3} - 2 \frac{1}{3 I \sqrt{3} - 3}$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\ z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.10 Problem.

Check the Zero Sum theorem for the following function

$$\frac{1}{z^6+1}$$

Hint.

$$\textit{no\_hint}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{1}{z^6+1}$$

We find singularities

$$[\{z = -I\},\ \{z = -\frac{1}{2}\sqrt{2-2\,I\,\sqrt{3}}\},\ \{z = \frac{1}{2}\sqrt{2-2\,I\,\sqrt{3}}\},\ \{z = -\frac{1}{2}\sqrt{2+2\,I\,\sqrt{3}}\},$$
$$\{z = \frac{1}{2}\sqrt{2+2\,I\,\sqrt{3}}\},\ \{z = I\}]$$

The singularity

$$z = -I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ -I) = \frac{1}{6}\,I$$

The singularity

$$z = -\frac{1}{2}\sqrt{2-2\,I\,\sqrt{3}}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ -\frac{1}{2}\sqrt{2-2\,I\,\sqrt{3}}) = -\frac{16}{3}\,\frac{1}{(2-2\,I\,\sqrt{3})^{(5/2)}}$$

The singularity

$$z = \frac{1}{2}\sqrt{2-2\,I\,\sqrt{3}}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ \frac{1}{2}\sqrt{2-2\,I\,\sqrt{3}}) = \frac{16}{3}\,\frac{1}{(2-2\,I\,\sqrt{3})^{(5/2)}}$$

The singularity

$$z = -\frac{1}{2}\sqrt{2 + 2\,I\,\sqrt{3}}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, -\frac{1}{2}\sqrt{2 + 2\,I\,\sqrt{3}}) = -\frac{16}{3}\,\frac{1}{(2 + 2\,I\,\sqrt{3})^{(5/2)}}$$

The singularity

$$z = \frac{1}{2}\sqrt{2 + 2\,I\,\sqrt{3}}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, \frac{1}{2}\sqrt{2 + 2\,I\,\sqrt{3}}) = \frac{16}{3}\,\frac{1}{(2 + 2\,I\,\sqrt{3})^{(5/2)}}$$

The singularity

$$z = I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, I) = -\frac{1}{6}\,I$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\, \infty) = 0$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\, z) = 0$$

Info.

$$\mathit{not\_given}$$

Comment.

$$\mathit{no\_comment}$$

2.11 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z}{(z^2 + 4\,z + 13)^2}$$

Hint.

$$\mathit{no\_hint}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z}{(z^2 + 4\,z + 13)^2}$$

We find singularities

$$[\{z = -2 - 3\,I\},\, \{z = -2 + 3\,I\}]$$

The singularity

$$z = -2 - 3\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, -2 - 3\,I) = -\frac{1}{54}\,I$$

The singularity

$$z = -2 + 3\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, -2 + 3\,I) = \frac{1}{54}\,I$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\, \infty) = 0$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\, z) = 0$$

Info.

$$\mathit{not\_given}$$

Comment.

$$\mathit{no\_comment}$$

2.12 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z^2}{(z^2+1)(z^2+9)}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$f(z) = \frac{z^2}{(z^2+1)(z^2+9)}$$

We find singularities

$$[\{z = -3\,I\}, \{z = -I\}, \{z = I\}, \{z = 3\,I\}]$$

The singularity

$$z = -3\,I$$

adds the following residue

$$\operatorname{res}(f(z), -3\,I) = \frac{3}{16}\,I$$

The singularity

$$z = -I$$

adds the following residue

$$\operatorname{res}(f(z), -I) = -\frac{1}{16}\,I$$

The singularity

$$z = I$$

adds the following residue

$$\operatorname{res}(f(z), I) = \frac{1}{16}\,I$$

The singularity

$$z = 3\,I$$

adds the following residue

$$\operatorname{res}(f(z), 3\,I) = -\frac{3}{16}\,I$$

At infinity we get the residue

$$\operatorname{res}(\mathrm{f}(z),\, \infty) = 0$$

and finally we obtain the sum

$$\sum \operatorname{res}(\mathrm{f}(z),\, z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.13 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z^2}{(z^2+1)^3}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z^2}{(z^2+1)^3}$$

We find singularities

$$[\{z = -I\}, \{z = I\}]$$

The singularity

$$z = -I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), -I) = \frac{1}{16} I$$

The singularity

$$z = I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), I) = -\frac{1}{16} I$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z), \infty) = 0$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z), z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.14 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z^2}{(z^2+4)^2}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$f(z) = \frac{z^2}{(z^2+4)^2}$$

We find singularities

$$[\{z = 2\,I\}, \{z = -2\,I\}]$$

The singularity

$$z = 2\,I$$

adds the following residue

$$\mathrm{res}(f(z), 2\,I) = -\frac{1}{8}\,I$$

The singularity

$$z = -2\,I$$

adds the following residue

$$\mathrm{res}(f(z), -2\,I) = \frac{1}{8}\,I$$

At infinity we get the residue

$$\mathrm{res}(f(z), \infty) = 0$$

and finally we obtain the sum

$$\sum \mathrm{res}(f(z), z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.15 Problem.

Check the Zero Sum theorem for the following function

$$\frac{(z^3+5\,z)\,e^{(I\,z)}}{z^4+10\,z^2+9}$$

Hint.

$$\textit{no\_hint}$$

Solution.
We denote

$$\mathrm{f}(z)=\frac{(z^3+5\,z)\,e^{(I\,z)}}{z^4+10\,z^2+9}$$

We find singularities

$$[\{z=-3\,I\},\,\{z=-I\},\,\{z=I\},\,\{z=3\,I\}]$$

The singularity

$$z=-3\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,-3\,I)=\frac{1}{4}\,e^3$$

The singularity

$$z=-I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,-I)=\frac{1}{4}\,e$$

The singularity

$$z=I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,I)=\frac{1}{4}\,e^{(-1)}$$

The singularity

$$z=3\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,3\,I)=\frac{1}{4}\,e^{(-3)}$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\,\infty) = -\frac{1}{4}\,e^3 - \frac{1}{4}\,e - \frac{1}{4}\,e^{(-1)} - \frac{1}{4}\,e^{(-3)}$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\,z) = 0$$

Info.

not_given

Comment.

no_comment

2.16 Problem.

Check the Zero Sum theorem for the following function

$$\frac{e^{(I\,z)}\,(z^2-1)}{z\,(z^2+1)}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z)=\frac{e^{(I\,z)}\,(z^2-1)}{z\,(z^2+1)}$$

We find singularities

$$[\{z=0\},\,\{z=-I\},\,\{z=I\}]$$

The singularity

$$z=0$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,0)=-1$$

The singularity

$$z=-I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,-I)=e$$

The singularity

$$z=I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,I)=e^{(-1)}$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\,\infty)=1-e-e^{(-1)}$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\,z)=0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.17 Problem.

Check the Zero Sum theorem for the following function

$$\frac{e^{(I\,z)}}{(z^2+4)\,(z-1)}$$

Hint.

$$\mathit{no\_hint}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{(z^2+4)\,(z-1)}$$

We find singularities

$$[\{z=1\},\ \{z=2\,I\},\ \{z=-2\,I\}]$$

The singularity

$$z=1$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ 1) = \frac{1}{5}\,e^I$$

The singularity

$$z=2\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ 2\,I) = (-\frac{1}{10}+\frac{1}{20}\,I)\,e^{(-2)}$$

The singularity

$$z=-2\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ -2\,I) = (-\frac{1}{10}-\frac{1}{20}\,I)\,e^2$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\ \infty) = -\frac{1}{5}\,e^I + (\frac{1}{10}-\frac{1}{20}\,I)\,e^{(-2)} + (\frac{1}{10}+\frac{1}{20}\,I)\,e^2$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\ z) = 0$$

Info.

not_given

Comment.

no_comment

2.18 Problem.

Check the Zero Sum theorem for the following function

$$\frac{e^{(I\,z)}}{z\,(z^2+1)}$$

Hint.

$$\mathit{no\_hint}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{z\,(z^2+1)}$$

We find singularities

$$[\{z=0\},\, \{z=-I\},\, \{z=I\}]$$

The singularity

$$z = 0$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, 0) = 1$$

The singularity

$$z = -I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, -I) = -\frac{1}{2}\,e$$

The singularity

$$z = I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, I) = -\frac{1}{2}\,e^{(-1)}$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\, \infty) = -1 + \frac{1}{2}\,e + \frac{1}{2}\,e^{(-1)}$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\, z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.19 Problem.

Check the Zero Sum theorem for the following function

$$\frac{e^{(I\,z)}}{z\,(z^2+9)^2}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z)=\frac{e^{(I\,z)}}{z\,(z^2+9)^2}$$

We find singularities

$$[\{z=-3\,I\},\,\{z=0\},\,\{z=3\,I\}]$$

The singularity

$$z=-3\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,-3\,I)=\frac{1}{324}\,e^3$$

The singularity

$$z=0$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,0)=\frac{1}{81}$$

The singularity

$$z=3\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,3\,I)=-\frac{5}{324}\,e^{(-3)}$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\,\infty)=-\frac{1}{324}\,e^3-\frac{1}{81}+\frac{5}{324}\,e^{(-3)}$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\,z)=0$$

Info.

not_given

Comment.

no_comment

2.20 Problem.

Check the Zero Sum theorem for the following function

$$\frac{e^{(I\,z)}}{z\,(z^2-2\,z+2)}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z)=\frac{e^{(I\,z)}}{z\,(z^2-2\,z+2)}$$

We find singularities

$$[\{z=0\},\,\{z=1-I\},\,\{z=1+I\}]$$

The singularity

$$z=0$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,0)=\frac{1}{2}$$

The singularity

$$z=1-I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,1-I)=(-\frac{1}{4}+\frac{1}{4}\,I)\,e^I\,e$$

The singularity

$$z=1+I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,1+I)=(-\frac{1}{4}-\frac{1}{4}\,I)\,e^I\,e^{(-1)}$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\,\infty)=-\frac{1}{2}+(\frac{1}{4}-\frac{1}{4}\,I)\,e^I\,e+(\frac{1}{4}+\frac{1}{4}\,I)\,e^I\,e^{(-1)}$$

and finally we obtain the sum

$$\sum\mathrm{res}(\mathrm{f}(z),\,z)=0$$

Info.

not_given

Comment.

no_comment

2.21 Problem.

Check the Zero Sum theorem for the following function

$$\frac{e^{(I\,z)}}{z^2+1}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{z^2+1}$$

We find singularities

$$[\{z=-I\},\,\{z=I\}]$$

The singularity

$$z=-I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,-I) = \frac{1}{2}\,I\,e$$

The singularity

$$z=I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,I) = -\frac{1}{2}\,I\,e^{(-1)}$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\,\infty) = -\frac{1}{2}\,I\,e + \frac{1}{2}\,I\,e^{(-1)}$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\,z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.22 Problem.

Check the Zero Sum theorem for the following function

$$\frac{e^{(I\,z)}}{z^2+4\,z+20}$$

Hint.

no_hint

Solution.
We denote

$$\mathrm{f}(z)=\frac{e^{(I\,z)}}{z^2+4\,z+20}$$

We find singularities

$$[\{z=-2-4\,I\},\,\{z=-2+4\,I\}]$$

The singularity

$$z=-2-4\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,-2-4\,I)=\frac{1}{8}\,\frac{I\,e^4}{(e^I)^2}$$

The singularity

$$z=-2+4\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,-2+4\,I)=-\frac{1}{8}\,\frac{I\,e^{(-4)}}{(e^I)^2}$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\,\infty)=-\frac{1}{8}\,\frac{I\,e^4}{(e^I)^2}+\frac{1}{8}\,\frac{I\,e^{(-4)}}{(e^I)^2}$$

and finally we obtain the sum

$$\sum\mathrm{res}(\mathrm{f}(z),\,z)=0$$

Info.

not_given

Comment.

no_comment

2.23 Problem.

Check the Zero Sum theorem for the following function

$$\frac{e^{(I\,z)}}{z^2 - 5\,z + 6}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{z^2 - 5\,z + 6}$$

We find singularities

$$[\{z = 2\}, \{z = 3\}]$$

The singularity

$$z = 2$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), 2) = -(e^I)^2$$

The singularity

$$z = 3$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), 3) = (e^I)^3$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z), \infty) = (e^I)^2 - (e^I)^3$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z), z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.24 Problem.

Check the Zero Sum theorem for the following function

$$\frac{e^{(I\,z)}}{z^3+1}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{z^3+1}$$

We find singularities

$$[\{z=-1\}, \{z=\frac{1}{2}-\frac{1}{2}\,I\,\sqrt{3}\}, \{z=\frac{1}{2}+\frac{1}{2}\,I\,\sqrt{3}\}]$$

The singularity

$$z=-1$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,-1) = \frac{1}{3}\,\frac{1}{e^I}$$

The singularity

$$z=\frac{1}{2}-\frac{1}{2}\,I\,\sqrt{3}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,\frac{1}{2}-\frac{1}{2}\,I\,\sqrt{3}) = -2\,\frac{(-1)^{(1/2\,\frac{1}{\pi})}\,\sqrt{e^{(\sqrt{3})}}}{3\,I\,\sqrt{3}+3}$$

The singularity

$$z=\frac{1}{2}+\frac{1}{2}\,I\,\sqrt{3}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,\frac{1}{2}+\frac{1}{2}\,I\,\sqrt{3}) = 2\,\frac{(-1)^{(1/2\,\frac{1}{\pi})}}{3\,I\,\sqrt{e^{(\sqrt{3})}}\,\sqrt{3}-3\,\sqrt{e^{(\sqrt{3})}}}$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\,\infty) = -\frac{1}{3}\,\frac{1}{e^I}+2\,\frac{(-1)^{(1/2\,\frac{1}{\pi})}\,\sqrt{e^{(\sqrt{3})}}}{3\,I\,\sqrt{3}+3}-2\,\frac{(-1)^{(1/2\,\frac{1}{\pi})}}{3\,I\,\sqrt{e^{(\sqrt{3})}}\,\sqrt{3}-3\,\sqrt{e^{(\sqrt{3})}}}$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\ z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.25 Problem.

Check the Zero Sum theorem for the following function

$$\frac{e^{(I\,z)}}{z^4 - 1}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{z^4 - 1}$$

We find singularities

$$[\{z = 1\},\, \{z = -1\},\, \{z = -I\},\, \{z = I\}]$$

The singularity

$$z = 1$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, 1) = \frac{1}{4}\, e^I$$

The singularity

$$z = -1$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, -1) = -\frac{1}{4}\, \frac{1}{e^I}$$

The singularity

$$z = -I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, -I) = -\frac{1}{4}\, I\, e$$

The singularity

$$z = I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, I) = \frac{1}{4}\, I\, e^{(-1)}$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\ \infty) = -\frac{1}{4}\,e^{I} + \frac{1}{4}\,\frac{1}{e^{I}} + \frac{1}{4}\,I\,e - \frac{1}{4}\,I\,e^{(-1)}$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\ z) = 0$$

Info.

not_given

Comment.

no_comment

2.26 Problem.

Check the Zero Sum theorem for the following function

$$\frac{e^{(I\,z)}}{z}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{z}$$

We find singularities

$$[\{z = 0\}]$$

The singularity

$$z = 0$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, 0) = 1$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\, \infty) = -1$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\, z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.27 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z\,e^{(I\,z)}}{1-z^4}$$

Hint.

$$\mathit{no\_hint}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z\,e^{(I\,z)}}{1-z^4}$$

We find singularities

$$[\{z=1\},\,\{z=-1\},\,\{z=-I\},\,\{z=I\}]$$

The singularity

$$z = 1$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,1) = -\frac{1}{4}\,e^{I}$$

The singularity

$$z = -1$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,-1) = -\frac{1}{4}\,\frac{1}{e^{I}}$$

The singularity

$$z = -I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,-I) = \frac{1}{4}\,e$$

The singularity

$$z = I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,I) = \frac{1}{4}\,e^{(-1)}$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\,\infty) = \frac{1}{4}\,e^{I} + \frac{1}{4}\,\frac{1}{e^{I}} - \frac{1}{4}\,e - \frac{1}{4}\,e^{(-1)}$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\,z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.28 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z\,e^{(I\,z)}}{z^2+4\,z+20}$$

Hint.

no_hint

Solution.
We denote

$$\mathrm{f}(z) = \frac{z\,e^{(I\,z)}}{z^2+4\,z+20}$$

We find singularities

$$[\{z=-2-4\,I\},\,\{z=-2+4\,I\}]$$

The singularity

$$z=-2-4\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,-2-4\,I) = -\frac{1}{8}\,\frac{I\,(4\,I\,e^4+2\,e^4)}{(e^I)^2}$$

The singularity

$$z=-2+4\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,-2+4\,I) = -\frac{1}{8}\,\frac{I\,(4\,I\,e^{(-4)}-2\,e^{(-4)})}{(e^I)^2}$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\,\infty) = \frac{1}{8}\,\frac{I\,(4\,I\,e^4+2\,e^4)}{(e^I)^2}+\frac{1}{8}\,\frac{I\,(4\,I\,e^{(-4)}-2\,e^{(-4)})}{(e^I)^2}$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\,z)=0$$

Info.

not_given

Comment.

no_comment

2.29 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z\,e^{(I\,z)}}{z^2+9}$$

Hint.

$$\mathit{no\_hint}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z\,e^{(I\,z)}}{z^2+9}$$

We find singularities

$$[\{z = -3\,I\},\,\{z = 3\,I\}]$$

The singularity

$$z = -3\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,-3\,I) = \frac{1}{2}\,e^3$$

The singularity

$$z = 3\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,3\,I) = \frac{1}{2}\,e^{(-3)}$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\,\infty) = -\frac{1}{2}\,e^3 - \frac{1}{2}\,e^{(-3)}$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\,z) = 0$$

Info.

$$\mathit{not\_given}$$

Comment.

$$\mathit{no\_comment}$$

2.30 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z\, e^{(I\, z)}}{z^2 - 2\, z + 10}$$

Hint.

$$\textit{no\_hint}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z\, e^{(I\, z)}}{z^2 - 2\, z + 10}$$

We find singularities

$$[\{z = 1 - 3\, I\},\ \{z = 1 + 3\, I\}]$$

The singularity

$$z = 1 - 3\, I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ 1 - 3\, I) = -\frac{1}{6}\, I\, (3\, I\, e^I\, e^3 - e^I\, e^3)$$

The singularity

$$z = 1 + 3\, I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ 1 + 3\, I) = -\frac{1}{6}\, I\, (3\, I\, e^I\, e^{(-3)} + e^I\, e^{(-3)})$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\ \infty) = \frac{1}{6}\, I\, (3\, I\, e^I\, e^3 - e^I\, e^3) + \frac{1}{6}\, I\, (3\, I\, e^I\, e^{(-3)} + e^I\, e^{(-3)})$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\ z) = 0$$

Info.

$$\textit{not\_given}$$

Comment.

$$\textit{no\_comment}$$

2.31 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z\,e^{(I\,z)}}{z^2-5\,z+6}$$

Hint.

$$\textit{no\_hint}$$

Solution.
We denote

$$\mathrm{f}(z)=\frac{z\,e^{(I\,z)}}{z^2-5\,z+6}$$

We find singularities

$$[\{z=2\},\,\{z=3\}]$$

The singularity

$$z=2$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,2)=-2\,(e^I)^2$$

The singularity

$$z=3$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,3)=3\,(e^I)^3$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\,\infty)=2\,(e^I)^2-3\,(e^I)^3$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\,z)=0$$

Info.

$$\textit{not\_given}$$

Comment.

$$\textit{no\_comment}$$

2.32 Problem.

Check the Zero Sum theorem for the following function

$$\frac{e^{(I\,z)}}{z^2}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^{(I\,z)}}{z^2}$$

We find singularities

$$[\{z = 0\}]$$

The singularity

$$z = 0$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, 0) = I$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\, \infty) = -I$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\, z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.33 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z^3\, e^{(I\, z)}}{z^4 + 5\, z^2 + 4}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z^3\, e^{(I\, z)}}{z^4 + 5\, z^2 + 4}$$

We find singularities

$$[\{z = -I\},\, \{z = 2\, I\},\, \{z = -2\, I\},\, \{z = I\}]$$

The singularity

$$z = -I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, -I) = -\frac{1}{6}\, e$$

The singularity

$$z = 2\, I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, 2\, I) = \frac{2}{3}\, e^{(-2)}$$

The singularity

$$z = -2\, I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, -2\, I) = \frac{2}{3}\, e^2$$

The singularity

$$z = I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\, I) = -\frac{1}{6}\, e^{(-1)}$$

At infinity we get the residue

$$\operatorname{res}(\mathrm{f}(z),\, \infty) = \frac{1}{6}\, e - \frac{2}{3}\, e^{(-2)} - \frac{2}{3}\, e^{2} + \frac{1}{6}\, e^{(-1)}$$

and finally we obtain the sum

$$\sum \operatorname{res}(\mathrm{f}(z),\, z) = 0$$

Info.

not_given

Comment.

no_comment

2.34 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z^2+1}{e^z}$$

Hint.

no_hint

Solution.
We denote

$$\mathrm{f}(z) = \frac{z^2+1}{e^z}$$

We find singularities

$$[\{z = -\infty\}]$$

At infinity we get the residue

$$\operatorname{res}(\mathrm{f}(z),\, \infty) = 0$$

and finally we obtain the sum

$$\sum \operatorname{res}(\mathrm{f}(z),\, z) = 0$$

Info.

not_given

Comment.

no_comment

2.35 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z^2+z-1}{z^2\,(z-1)}$$

Hint.

$$\textit{no\_hint}$$

Solution.
We denote

$$f(z) = \frac{z^2+z-1}{z^2\,(z-1)}$$

We find singularities

$$[\{z=1\},\,\{z=0\}]$$

The singularity

$$z = 1$$

adds the following residue

$$\mathrm{res}(f(z),\,1) = 1$$

The singularity

$$z = 0$$

adds the following residue

$$\mathrm{res}(f(z),\,0) = 0$$

At infinity we get the residue

$$\mathrm{res}(f(z),\,\infty) = -1$$

and finally we obtain the sum

$$\sum \mathrm{res}(f(z),\,z) = 0$$

Info.

$$\textit{not\_given}$$

Comment.

$$\textit{no\_comment}$$

2.36 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z^2 - 2z + 5}{(z-2)(z^2+1)}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$f(z) = \frac{z^2 - 2z + 5}{(z-2)(z^2+1)}$$

We find singularities

$$[\{z = 2\}, \{z = -I\}, \{z = I\}]$$

The singularity

$$z = 2$$

adds the following residue

$$\mathrm{res}(f(z), 2) = 1$$

The singularity

$$z = -I$$

adds the following residue

$$\mathrm{res}(f(z), -I) = -I$$

The singularity

$$z = I$$

adds the following residue

$$\mathrm{res}(f(z), I) = I$$

At infinity we get the residue

$$\mathrm{res}(f(z), \infty) = -1$$

and finally we obtain the sum

$$\sum \mathrm{res}(f(z), z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.37 Problem.

Check the Zero Sum theorem for the following function

$$\frac{1}{(z+1)(z-1)}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$f(z) = \frac{1}{(z+1)(z-1)}$$

We find singularities

$$[\{z = 1\}, \{z = -1\}]$$

The singularity

$$z = 1$$

adds the following residue

$$\operatorname{res}(f(z), 1) = \frac{1}{2}$$

The singularity

$$z = -1$$

adds the following residue

$$\operatorname{res}(f(z), -1) = \frac{-1}{2}$$

At infinity we get the residue

$$\operatorname{res}(f(z), \infty) = 0$$

and finally we obtain the sum

$$\sum \operatorname{res}(f(z), z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.38 Problem.

Check the Zero Sum theorem for the following function

$$\frac{1}{z\,(1-z^2)}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{1}{z\,(1-z^2)}$$

We find singularities

$$[\{z=1\},\ \{z=-1\},\ \{z=0\}]$$

The singularity

$$z = 1$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ 1) = \frac{-1}{2}$$

The singularity

$$z = -1$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ -1) = \frac{-1}{2}$$

The singularity

$$z = 0$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ 0) = 1$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\ \infty) = 0$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\ z) = 0$$

Info.

not_given

Comment.

no_comment

2.39 Problem.

Check the Zero Sum theorem for the following function

$$\frac{1}{z\,(z^2+4)^2}$$

Hint.

$$\textit{no\_hint}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{1}{z\,(z^2+4)^2}$$

We find singularities

$$[\{z=0\},\,\{z=2\,I\},\,\{z=-2\,I\}]$$

The singularity

$$z=0$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,0) = \frac{1}{16}$$

The singularity

$$z=2\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,2\,I) = \frac{-1}{32}$$

The singularity

$$z=-2\,I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\,-2\,I) = \frac{-1}{32}$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\,\infty) = 0$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\,z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.40 Problem.

Check the Zero Sum theorem for the following function

$$\frac{1}{z\,(z-1)}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{1}{z\,(z-1)}$$

We find singularities

$$[\{z = 1\},\ \{z = 0\}]$$

The singularity

$$z = 1$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ 1) = 1$$

The singularity

$$z = 0$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ 0) = -1$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z),\ \infty) = 0$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z),\ z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.41 Problem.

Check the Zero Sum theorem for the following function

$$\frac{1}{z(z-1)}$$

Hint.

$$\textit{no\_hint}$$

Solution.
We denote

$$f(z) = \frac{1}{z(z-1)}$$

We find singularities

$$[\{z = 1\}, \{z = 0\}]$$

The singularity

$$z = 1$$

adds the following residue

$$\mathrm{res}(f(z), 1) = 1$$

The singularity

$$z = 0$$

adds the following residue

$$\mathrm{res}(f(z), 0) = -1$$

At infinity we get the residue

$$\mathrm{res}(f(z), \infty) = 0$$

and finally we obtain the sum

$$\sum \mathrm{res}(f(z), z) = 0$$

Info.

$$\textit{not\_given}$$

Comment.

$$\textit{no\_comment}$$

2.42 Problem.

Check the Zero Sum theorem for the following function

$$\frac{1}{(z^2+1)^2}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$f(z) = \frac{1}{(z^2+1)^2}$$

We find singularities

$$[\{z = -I\}, \{z = I\}]$$

The singularity

$$z = -I$$

adds the following residue

$$\mathrm{res}(f(z), -I) = \frac{1}{4} I$$

The singularity

$$z = I$$

adds the following residue

$$\mathrm{res}(f(z), I) = -\frac{1}{4} I$$

At infinity we get the residue

$$\mathrm{res}(f(z), \infty) = 0$$

and finally we obtain the sum

$$\sum \mathrm{res}(f(z), z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.43 Problem.

Check the Zero Sum theorem for the following function

$$\frac{1}{z^3 - z^5}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{1}{z^3 - z^5}$$

We find singularities

$$[\{z = 1\}, \{z = -1\}, \{z = 0\}]$$

The singularity

$$z = 1$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), 1) = \frac{-1}{2}$$

The singularity

$$z = -1$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), -1) = \frac{-1}{2}$$

The singularity

$$z = 0$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), 0) = 1$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z), \infty) = 0$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z), z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.44 Problem.

Check the Zero Sum theorem for the following function

$$\frac{1}{z^4+1}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{1}{z^4+1}$$

We find singularities

$[\{z = \frac{1}{2}\sqrt{2}+\frac{1}{2} I\sqrt{2}\}, \{z = -\frac{1}{2}\sqrt{2}-\frac{1}{2} I\sqrt{2}\}, \{z = \frac{1}{2}\sqrt{2}-\frac{1}{2} I\sqrt{2}\}, \{z = -\frac{1}{2}\sqrt{2}+\frac{1}{2} I\sqrt{2}\}]$

The singularity

$$z = \frac{1}{2}\sqrt{2} + \frac{1}{2} I\sqrt{2}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ \frac{1}{2}\sqrt{2} + \frac{1}{2} I\sqrt{2}) = \frac{1}{2 I\sqrt{2} - 2\sqrt{2}}$$

The singularity

$$z = -\frac{1}{2}\sqrt{2} - \frac{1}{2} I\sqrt{2}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ -\frac{1}{2}\sqrt{2} - \frac{1}{2} I\sqrt{2}) = -\frac{1}{2 I\sqrt{2} - 2\sqrt{2}}$$

The singularity

$$z = \frac{1}{2}\sqrt{2} - \frac{1}{2} I\sqrt{2}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ \frac{1}{2}\sqrt{2} - \frac{1}{2} I\sqrt{2}) = -\frac{1}{2 I\sqrt{2} + 2\sqrt{2}}$$

The singularity

$$z = -\frac{1}{2}\sqrt{2} + \frac{1}{2} I\sqrt{2}$$

adds the following residue

$$\operatorname{res}(\mathrm{f}(z),\ -\frac{1}{2}\sqrt{2}+\frac{1}{2}\,I\,\sqrt{2}) = \frac{1}{2\,I\,\sqrt{2}+2\,\sqrt{2}}$$

At infinity we get the residue

$$\operatorname{res}(\mathrm{f}(z),\ \infty) = 0$$

and finally we obtain the sum

$$\sum \operatorname{res}(\mathrm{f}(z),\ z) = 0$$

Info.

not_given

Comment.

no_comment

2.45 Problem.

Check the Zero Sum theorem for the following function

$$\frac{1}{z - z^3}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{1}{z - z^3}$$

We find singularities

$$[\{z = 1\}, \{z = -1\}, \{z = 0\}]$$

The singularity

$$z = 1$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), 1) = \frac{-1}{2}$$

The singularity

$$z = -1$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), -1) = \frac{-1}{2}$$

The singularity

$$z = 0$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), 0) = 1$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z), \infty) = 0$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z), z) = 0$$

Info.

not_given

Comment.

no_comment

2.46 Problem.

Check the Zero Sum theorem for the following function

$$\frac{1}{z}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$f(z) = \frac{1}{z}$$

We find singularities

$$[\{z = 0\}]$$

The singularity

$$z = 0$$

adds the following residue

$$\mathrm{res}(f(z), 0) = 1$$

At infinity we get the residue

$$\mathrm{res}(f(z), \infty) = -1$$

and finally we obtain the sum

$$\sum \mathrm{res}(f(z), z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.47 Problem.

Check the Zero Sum theorem for the following function

$$\frac{e^z}{(z^2+9)\,z^2}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$f(z) = \frac{e^z}{(z^2+9)\,z^2}$$

We find singularities

$$[\{z=\infty\},\,\{z=-3\,I\},\,\{z=0\},\,\{z=3\,I\}]$$

The singularity

$$z = -3\,I$$

adds the following residue

$$\mathrm{res}(f(z),\,-3\,I) = -\frac{1}{54}\,\frac{I}{(e^I)^3}$$

The singularity

$$z = 0$$

adds the following residue

$$\mathrm{res}(f(z),\,0) = \frac{1}{9}$$

The singularity

$$z = 3\,I$$

adds the following residue

$$\mathrm{res}(f(z),\,3\,I) = \frac{1}{54}\,I\,(e^I)^3$$

At infinity we get the residue

$$\mathrm{res}(f(z),\,\infty) = \frac{1}{54}\,\frac{I}{(e^I)^3} - \frac{1}{9} - \frac{1}{54}\,I\,(e^I)^3$$

and finally we obtain the sum

$$\sum \mathrm{res}(f(z),\,z) = 0$$

Info.

not_given

Comment.

no_comment

2.48 Problem.

Check the Zero Sum theorem for the following function

$$\frac{e^z}{z^2+1}$$

Hint.

$$\mathit{no\_hint}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{e^z}{z^2+1}$$

We find singularities

$$[\{z = \infty\}, \{z = -I\}, \{z = I\}]$$

The singularity

$$z = -I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), -I) = \frac{1}{2} \frac{I}{e^I}$$

The singularity

$$z = I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), I) = -\frac{1}{2} I e^I$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z), \infty) = -\frac{1}{2} \frac{I}{e^I} + \frac{1}{2} I e^I$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z), z) = 0$$

Info.

$$\mathit{not\_given}$$

Comment.

$$\mathit{no\_comment}$$

2.49 Problem.

Check the Zero Sum theorem for the following function

$$\frac{e^z}{(z^2+9)\,z^2}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$f(z) = \frac{e^z}{(z^2+9)\,z^2}$$

We find singularities

$$[\{z=\infty\},\,\{z=-3\,I\},\,\{z=0\},\,\{z=3\,I\}]$$

The singularity

$$z=-3\,I$$

adds the following residue

$$\mathrm{res}(f(z),\,-3\,I) = -\frac{1}{54}\,\frac{I}{(e^I)^3}$$

The singularity

$$z=0$$

adds the following residue

$$\mathrm{res}(f(z),\,0) = \frac{1}{9}$$

The singularity

$$z=3\,I$$

adds the following residue

$$\mathrm{res}(f(z),\,3\,I) = \frac{1}{54}\,I\,(e^I)^3$$

At infinity we get the residue

$$\mathrm{res}(f(z),\,\infty) = \frac{1}{54}\,\frac{I}{(e^I)^3} - \frac{1}{9} - \frac{1}{54}\,I\,(e^I)^3$$

and finally we obtain the sum

$$\sum \mathrm{res}(f(z),\,z) = 0$$

Info.

not_given

Comment.

no_comment

2.50 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z}{(z-1)(z-2)^2}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$f(z) = \frac{z}{(z-1)(z-2)^2}$$

We find singularities

$$[\{z = 1\}, \{z = 2\}]$$

The singularity

$$z = 1$$

adds the following residue

$$\text{res}(f(z), 1) = 1$$

The singularity

$$z = 2$$

adds the following residue

$$\text{res}(f(z), 2) = -1$$

At infinity we get the residue

$$\text{res}(f(z), \infty) = 0$$

and finally we obtain the sum

$$\sum \text{res}(f(z), z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.51 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z^2}{(z^2+1)^2}$$

Hint.

$$no\_hint$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z^2}{(z^2+1)^2}$$

We find singularities

$$[\{z = -I\}, \{z = I\}]$$

The singularity

$$z = -I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), -I) = \frac{1}{4} I$$

The singularity

$$z = I$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), I) = -\frac{1}{4} I$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z), \infty) = 0$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z), z) = 0$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

2.52 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z^4}{z^4+1}$$

Hint.

$$\mathit{no\_hint}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z^4}{z^4+1}$$

We find singularities

$$[\{z = \frac{1}{2}\sqrt{2}+\frac{1}{2}I\sqrt{2}\}, \{z = -\frac{1}{2}\sqrt{2}-\frac{1}{2}I\sqrt{2}\}, \{z = \frac{1}{2}\sqrt{2}-\frac{1}{2}I\sqrt{2}\}, \{z = -\frac{1}{2}\sqrt{2}+\frac{1}{2}I\sqrt{2}\}]$$

The singularity

$$z = \frac{1}{2}\sqrt{2} + \frac{1}{2}I\sqrt{2}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ \frac{1}{2}\sqrt{2} + \frac{1}{2}I\sqrt{2}) = -\frac{1}{2I\sqrt{2} - 2\sqrt{2}}$$

The singularity

$$z = -\frac{1}{2}\sqrt{2} - \frac{1}{2}I\sqrt{2}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ -\frac{1}{2}\sqrt{2} - \frac{1}{2}I\sqrt{2}) = \frac{1}{2I\sqrt{2} - 2\sqrt{2}}$$

The singularity

$$z = \frac{1}{2}\sqrt{2} - \frac{1}{2}I\sqrt{2}$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z),\ \frac{1}{2}\sqrt{2} - \frac{1}{2}I\sqrt{2}) = \frac{1}{2I\sqrt{2} + 2\sqrt{2}}$$

The singularity

$$z = -\frac{1}{2}\sqrt{2} + \frac{1}{2}I\sqrt{2}$$

adds the following residue

$$\operatorname{res}(\mathrm{f}(z),\, -\frac{1}{2}\sqrt{2} + \frac{1}{2}\, I\, \sqrt{2}) = -\frac{1}{2\, I\, \sqrt{2} + 2\, \sqrt{2}}$$

At infinity we get the residue

$$\operatorname{res}(\mathrm{f}(z),\, \infty) = 0$$

and finally we obtain the sum

$$\sum \operatorname{res}(\mathrm{f}(z),\, z) = 0$$

Info.

not_given

Comment.

no_comment

2.53 Problem.

Check the Zero Sum theorem for the following function

$$\frac{z^5}{(1-z)^2}$$

Hint.

$$\mathit{no\_hint}$$

Solution.
We denote

$$\mathrm{f}(z) = \frac{z^5}{(1-z)^2}$$

We find singularities

$$[\{z = \infty\}, \{z = 1\}, \{z = -\infty\}]$$

The singularity

$$z = 1$$

adds the following residue

$$\mathrm{res}(\mathrm{f}(z), 1) = 5$$

At infinity we get the residue

$$\mathrm{res}(\mathrm{f}(z), \infty) = -5$$

and finally we obtain the sum

$$\sum \mathrm{res}(\mathrm{f}(z), z) = 0$$

Info.

$$\mathit{not\_given}$$

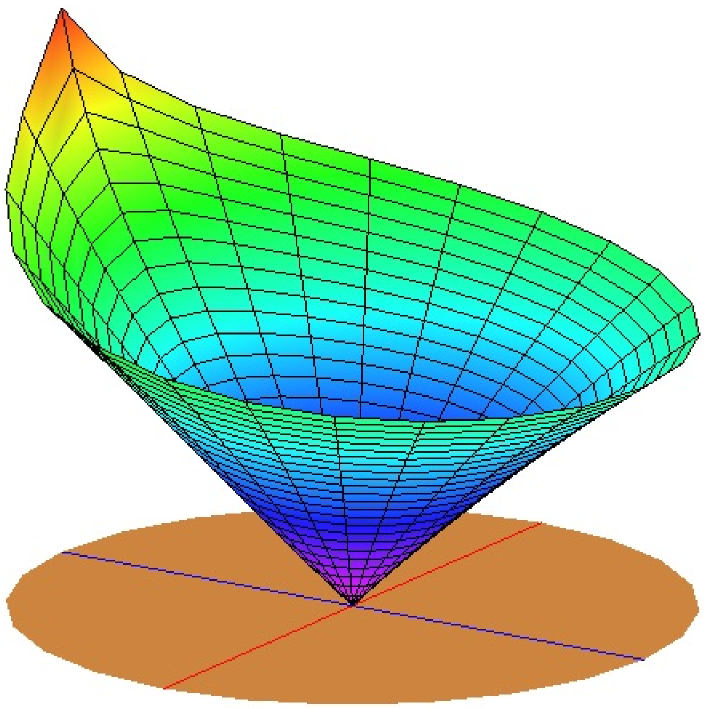
Comment.

$$\mathit{no\_comment}$$

3 Power series problems

Example

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}$$



3.1 Problem.

Sum the following power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}\, z^n}{n}$$

Hint.

$$derive\_once\_and\_sum$$

Solution.
We denote the n-th term in the series by

$$\mathrm{a}(n) = \frac{(-1)^{(n+1)}\, z^n}{n}$$

For the ratio test we need the term $a(n+1)$

$$\mathrm{a}(n+1) = \frac{(-1)^{(n+2)}\, z^{(n+1)}}{n+1}$$

Ratio test computes

$$\lim_{n\to\infty} \frac{|a(n+1)|}{|a(n)|}$$

and obtains in our case

$$\lim_{n\to\infty} \left| \frac{z^{(n+1)}\, n}{(n+1)\, z^n} \right| = |z|$$

Moreover we can check the root test computing

$$\lim_{n\to\infty} \sqrt[n]{|a(n)|}$$

and obtain in our case

$$\lim_{n\to\infty} [\frac{(-1)^{(n+1)}\, z^n}{n}]^{(\frac{1}{n})} = |z|$$

From this we conclude the radius of convergence R. For $|z| < R$ we sum the series using common tricks for power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}\, z^n}{n} = \ln(1+z)$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

3.2 Problem.

Sum the following power series

$$\sum_{n=1}^{\infty} (-1)^n\, n^2\, z^n$$

Hint.

$$divide\_by\_z\_and\_integrate$$

Solution.
We denote the n-th term in the series by

$$\mathrm{a}(n) = (-1)^n\, n^2\, z^n$$

For the ratio test we need the term $a(n+1)$

$$\mathrm{a}(n+1) = (-1)^{(n+1)}\, (n+1)^2\, z^{(n+1)}$$

Ratio test computes

$$\lim_{n\to\infty} \frac{|a(n+1)|}{|a(n)|}$$

and obtains in our case

$$\lim_{n\to\infty} \left| \frac{(n+1)^2\, z^{(n+1)}}{n^2\, z^n} \right| = |z|$$

Moreover we can check the root test computing

$$\lim_{n\to\infty} \sqrt[n]{|a(n)|}$$

and obtain in our case

$$\lim_{n\to\infty} [(-1)^n\, n^2\, z^n]^{(\frac{1}{n})} = |z|$$

From this we conclude the radius of convergence R. For $|z| < R$ we sum the series using common tricks for power series

$$\sum_{n=1}^{\infty} (-1)^n\, n^2\, z^n = -\frac{z\,(-z+1)}{(1+z)^3}$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

3.3 Problem.

Sum the following power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, n^3\, z^n}{(n+1)!}$$

Hint.

$$manipulate\_the\_numerator\_to\_cancel$$

Solution.
We denote the n-th term in the series by

$$\mathrm{a}(n) = \frac{(-1)^n\, n^3\, z^n}{(n+1)!}$$

For the ratio test we need the term $a(n+1)$

$$\mathrm{a}(n+1) = \frac{(-1)^{(n+1)}\, (n+1)^3\, z^{(n+1)}}{(n+2)!}$$

Ratio test computes

$$\lim_{n\to\infty} \frac{|a(n+1)|}{|a(n)|}$$

and obtains in our case

$$\lim_{n\to\infty} \left| \frac{(n+1)^3\, z^{(n+1)}\, (n+1)!}{(n+2)!\, n^3\, z^n} \right| = 0$$

Moreover we can check the root test computing

$$\lim_{n\to\infty} \sqrt[n]{|a(n)|}$$

and obtain in our case

$$\lim_{n\to\infty} [\frac{(-1)^n\, n^3\, z^n}{(n+1)!}]^{(\frac{1}{n})} = \left| \lim_{n\to\infty} (\frac{(-1)^n\, n^3\, z^n}{(n+1)!})^{(\frac{1}{n})} \right|$$

From this we conclude the radius of convergence R. For $|z| < R$ we sum the series using common tricks for power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, n^3\, z^n}{(n+1)!} =$$

$$-\frac{1}{2}\, z \left(2\, \frac{1 - e^{(-z)}\, (1+z)}{z^2} - 12\, \frac{1 - e^{(-z)}\, (1 + z + \frac{1}{2}\, z^2)}{z^2} + 12\, \frac{1 - e^{(-z)}\, (1 + z + \frac{1}{2}\, z^2 + \frac{1}{6}\, z^3)}{z^2} \right)$$

Info.

not_given

Comment.

no_comment

3.4 Problem.

Sum the following power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n(2n-1)}$$

Hint.

make_the_power_2n_and_derive_twice

Solution.
We denote the n-th term in the series by

$$\mathrm{a}(n) = \frac{(-1)^n z^n}{n(2n-1)}$$

For the ratio test we need the term $a(n+1)$

$$\mathrm{a}(n+1) = \frac{(-1)^{(n+1)} z^{(n+1)}}{(n+1)(2n+1)}$$

Ratio test computes

$$\lim_{n\to\infty} \frac{|a(n+1)|}{|a(n)|}$$

and obtains in our case

$$\lim_{n\to\infty} \left| \frac{z^{(n+1)} n (2n-1)}{(n+1)(2n+1) z^n} \right| = |z|$$

Moreover we can check the root test computing

$$\lim_{n\to\infty} \sqrt[n]{|a(n)|}$$

and obtain in our case

$$\lim_{n\to\infty} [\frac{(-1)^n z^n}{n(2n-1)}]^{(\frac{1}{n})} = |z|$$

From this we conclude the radius of convergence R. For $|z| < R$ we sum the series using common tricks for power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n(2n-1)} = -z \,\mathrm{hypergeom}([1,\, 1,\, \frac{1}{2}],\, [\frac{3}{2},\, 2],\, -z)$$

Info.

not_given

Comment.

no_comment

3.5 Problem.

Sum the following power series

$$\sum_{n=0}^{\infty} \frac{(2\,n+1)\,z^n}{n!}$$

Hint.

$$prepare\_a\_combination\_of\_exponentials$$

Solution.
We denote the n-th term in the series by

$$\mathrm{a}(n) = \frac{(2\,n+1)\,z^n}{n!}$$

For the ratio test we need the term $a(n+1)$

$$\mathrm{a}(n+1) = \frac{(2\,n+3)\,z^{(n+1)}}{(n+1)!}$$

Ratio test computes

$$\lim_{n\to\infty} \frac{|a(n+1)|}{|a(n)|}$$

and obtains in our case

$$\lim_{n\to\infty} \left| \frac{(2\,n+3)\,z^{(n+1)}\,n!}{(n+1)!\,(2\,n+1)\,z^n} \right| = 0$$

Moreover we can check the root test computing

$$\lim_{n\to\infty} \sqrt[n]{|a(n)|}$$

and obtain in our case

$$\lim_{n\to\infty} [\frac{(2\,n+1)\,z^n}{n!}]^{(\frac{1}{n})} = \left| \lim_{n\to\infty} (\frac{(2\,n+1)\,z^n}{n!})^{(\frac{1}{n})} \right|$$

From this we conclude the radius of convergence R. For $|z| < R$ we sum the series using common tricks for power series

$$\sum_{n=0}^{\infty} \frac{(2\,n+1)\,z^n}{n!} = 2\,e^z\,(\frac{1}{2} + z)$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

3.6 Problem.

Sum the following power series

$$\sum_{n=0}^{\infty} \frac{(n^2-2)\, z^n}{2^n\, n!}$$

Hint.

prepare_a_combination_of_exponentials

Solution.
We denote the n-th term in the series by

$$\mathrm{a}(n) = \frac{(n^2-2)\, z^n}{2^n\, n!}$$

For the ratio test we need the term $a(n+1)$

$$\mathrm{a}(n+1) = \frac{((n+1)^2-2)\, z^{(n+1)}}{2^{(n+1)}\,(n+1)!}$$

Ratio test computes

$$\lim_{n\to\infty} \frac{|a(n+1)|}{|a(n)|}$$

and obtains in our case

$$\lim_{n\to\infty} \frac{2^n \left| \frac{((n+1)^2-2)\, z^{(n+1)}\, n!}{(n+1)!\,(n^2-2)\, z^n} \right|}{2^{(n+1)}} = 0$$

Moreover we can check the root test computing

$$\lim_{n\to\infty} \sqrt[n]{|a(n)|}$$

and obtain in our case

$$\lim_{n\to\infty} [\frac{(n^2-2)\, z^n}{2^n\, n!}]^{(\frac{1}{n})} = \left| \lim_{n\to\infty} (\frac{(n^2-2)\, z^n}{2^n\, n!})^{(\frac{1}{n})} \right|$$

From this we conclude the radius of convergence R. For $|z| < R$ we sum the series using common tricks for power series

$$\sum_{n=0}^{\infty} \frac{(n^2-2)\, z^n}{2^n\, n!} = -2\, e^{(1/2\, z)} + \frac{1}{2}\sqrt{2}\, e^{(1/2\, z)}\, z + \frac{1}{2}(-\sqrt{2}+1)\, e^{(1/2\, z)}\, z + \frac{1}{4}\, e^{(1/2\, z)}\, z^2$$

Info.

not_given

Comment.

no_comment

3.7 Problem.

Sum the following power series

$$\sum_{n=0}^{\infty} n\,(n+1)\,z^n$$

Hint.

$$divide\_by\_z\_and\_integrate\_twice$$

Solution.
We denote the n-th term in the series by

$$\mathrm{a}(n) = n\,(n+1)\,z^n$$

For the ratio test we need the term $a(n+1)$

$$\mathrm{a}(n+1) = (n+1)\,(n+2)\,z^{(n+1)}$$

Ratio test computes

$$\lim_{n\to\infty} \frac{|a(n+1)|}{|a(n)|}$$

and obtains in our case

$$\lim_{n\to\infty} \left| \frac{(n+2)\,z^{(n+1)}}{n\,z^n} \right| = |z|$$

Moreover we can check the root test computing

$$\lim_{n\to\infty} \sqrt[n]{|a(n)|}$$

and obtain in our case

$$\lim_{n\to\infty} \left[n\,(n+1)\,z^n\right]^{\left(\frac{1}{n}\right)} = |z|$$

From this we conclude the radius of convergence R. For $|z| < R$ we sum the series using common tricks for power series

$$\sum_{n=0}^{\infty} n\,(n+1)\,z^n = -2\,\frac{z}{(z-1)^3}$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

3.8 Problem.

Sum the following power series

$$\sum_{n=0}^{\infty} \frac{n\,z^n}{5^n}$$

Hint.

$$divide\_by\_z\_and\_integrate$$

Solution.
We denote the n-th term in the series by

$$\mathrm{a}(n) = \frac{n\,z^n}{5^n}$$

For the ratio test we need the term $a(n+1)$

$$\mathrm{a}(n+1) = \frac{(n+1)\,z^{(n+1)}}{5^{(n+1)}}$$

Ratio test computes

$$\lim_{n\to\infty} \frac{|a(n+1)|}{|a(n)|}$$

and obtains in our case

$$\lim_{n\to\infty} \frac{5^n \left|\frac{(n+1)\,z^{(n+1)}}{n\,z^n}\right|}{5^{(n+1)}} = \frac{1}{5}\;|z|$$

Moreover we can check the root test computing

$$\lim_{n\to\infty} \sqrt[n]{|a(n)|}$$

and obtain in our case

$$\lim_{n\to\infty} [\frac{n\,z^n}{5^n}]^{(\frac{1}{n})} = \left|\lim_{n\to\infty} (\frac{n\,z^n}{5^n})^{(\frac{1}{n})}\right|$$

From this we conclude the radius of convergence R. For $|z| < R$ we sum the series using common tricks for power series

$$\sum_{n=0}^{\infty} \frac{n\,z^n}{5^n} = 5\,\frac{z}{(z-5)^2}$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

3.9 Problem.

Sum the following power series

$$\sum_{n=1}^{\infty} n^2 z^{(n-1)}$$

Hint.

$$integrate\_divide\_by\_z\_and\_integrate$$

Solution.
We denote the n-th term in the series by

$$\mathrm{a}(n) = n^2 z^{(n-1)}$$

For the ratio test we need the term $a(n+1)$

$$\mathrm{a}(n+1) = (n+1)^2 z^n$$

Ratio test computes

$$\lim_{n\to\infty} \frac{|a(n+1)|}{|a(n)|}$$

and obtains in our case

$$\lim_{n\to\infty} \left| \frac{(n+1)^2 z^n}{n^2 z^{(n-1)}} \right| = |z|$$

Moreover we can check the root test computing

$$\lim_{n\to\infty} \sqrt[n]{|a(n)|}$$

and obtain in our case

$$\lim_{n\to\infty} [n^2 z^{(n-1)}]^{(\frac{1}{n})} = |z|$$

From this we conclude the radius of convergence R. For $|z| < R$ we sum the series using common tricks for power series

$$\sum_{n=1}^{\infty} n^2 z^{(n-1)} = \frac{1+z}{(-z+1)^3}$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

3.10 Problem.

Sum the following power series

$$\sum_{n=0}^{\infty} \frac{z^{(2\,n)}}{(2\,n)!}$$

Hint.

$$manipulate\_to\_exponentials$$

Solution.
We denote the n-th term in the series by

$$\mathrm{a}(n) = \frac{z^{(2\,n)}}{(2\,n)!}$$

For the ratio test we need the term $a(n+1)$

$$\mathrm{a}(n+1) = \frac{z^{(2\,n+2)}}{(2\,n+2)!}$$

Ratio test computes

$$\lim_{n\to\infty} \frac{|a(n+1)|}{|a(n)|}$$

and obtains in our case

$$\lim_{n\to\infty} \left| \frac{z^{(2\,n+2)}\,(2\,n)!}{(2\,n+2)!\,z^{(2\,n)}} \right| = 0$$

Moreover we can check the root test computing

$$\lim_{n\to\infty} \sqrt[n]{|a(n)|}$$

and obtain in our case

$$\lim_{n\to\infty} [\frac{z^{(2\,n)}}{(2\,n)!}]^{(\frac{1}{n})} = 0$$

From this we conclude the radius of convergence R. For $|z| < R$ we sum the series using common tricks for power series

$$\sum_{n=0}^{\infty} \frac{z^{(2\,n)}}{(2\,n)!} = \cosh(z)$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

3.11 Problem.

Sum the following power series

$$\sum_{n=0}^{\infty} \frac{z^{(2\,n-1)}}{(2\,n-1)!}$$

Hint.

$$manipulate\_to\_exponentials$$

Solution.
We denote the n-th term in the series by

$$\mathrm{a}(n) = \frac{z^{(2\,n-1)}}{(2\,n-1)!}$$

For the ratio test we need the term $a(n+1)$

$$\mathrm{a}(n+1) = \frac{z^{(2\,n+1)}}{(2\,n+1)!}$$

Ratio test computes

$$\lim_{n\to\infty} \frac{|a(n+1)|}{|a(n)|}$$

and obtains in our case

$$\lim_{n\to\infty} \left| \frac{z^{(2\,n+1)}\,(2\,n-1)!}{(2\,n+1)!\,z^{(2\,n-1)}} \right| = 0$$

Moreover we can check the root test computing

$$\lim_{n\to\infty} \sqrt[n]{|a(n)|}$$

and obtain in our case

$$\lim_{n\to\infty} [\frac{z^{(2\,n-1)}}{(2\,n-1)!}]^{(\frac{1}{n})} = \left| \lim_{n\to\infty} (\frac{z^{(2\,n-1)}}{(2\,n-1)!})^{(\frac{1}{n})} \right|$$

From this we conclude the radius of convergence R. For $|z| < R$ we sum the series using common tricks for power series

$$\sum_{n=0}^{\infty} \frac{z^{(2\,n-1)}}{(2\,n-1)!} = \sinh(z)$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

3.12 Problem.

Sum the following power series

$$\sum_{n=1}^{\infty} \frac{z^n}{n\,(n+1)}$$

Hint.

$$multiple\_by\_z\_and\_derive\_twice$$

Solution.
We denote the n-th term in the series by

$$\mathrm{a}(n) = \frac{z^n}{n\,(n+1)}$$

For the ratio test we need the term $a(n+1)$

$$\mathrm{a}(n+1) = \frac{z^{(n+1)}}{(n+1)\,(n+2)}$$

Ratio test computes

$$\lim_{n\to\infty} \frac{|a(n+1)|}{|a(n)|}$$

and obtains in our case

$$\lim_{n\to\infty} \left| \frac{z^{(n+1)}\,n}{(n+2)\,z^n} \right| = |z|$$

Moreover we can check the root test computing

$$\lim_{n\to\infty} \sqrt[n]{|a(n)|}$$

and obtain in our case

$$\lim_{n\to\infty} [\frac{z^n}{n\,(n+1)}]^{(\frac{1}{n})} = |z|$$

From this we conclude the radius of convergence R. For $|z| < R$ we sum the series using common tricks for power series

$$\sum_{n=1}^{\infty} \frac{z^n}{n\,(n+1)} = \frac{1}{2}\,z \left(2\,\frac{(-\frac{\ln(-z+1)}{z} - 1)\,(z-1)}{z} - \frac{-2\,z+2}{z-1} \right)$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$

3.13 Problem.

Sum the following power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Hint.

$$manipulate\_to\_exponentials$$

Solution.
We denote the n-th term in the series by

$$\mathrm{a}(n) = \frac{z^n}{n!}$$

For the ratio test we need the term $a(n+1)$

$$\mathrm{a}(n+1) = \frac{z^{(n+1)}}{(n+1)!}$$

Ratio test computes

$$\lim_{n\to\infty} \frac{|a(n+1)|}{|a(n)|}$$

and obtains in our case

$$\lim_{n\to\infty} \left| \frac{z^{(n+1)}\, n!}{(n+1)!\, z^n} \right| = 0$$

Moreover we can check the root test computing

$$\lim_{n\to\infty} \sqrt[n]{|a(n)|}$$

and obtain in our case

$$\lim_{n\to\infty} [\frac{z^n}{n!}]^{(\frac{1}{n})} = \left| \lim_{n\to\infty} (\frac{z^n}{n!})^{(\frac{1}{n})} \right|$$

From this we conclude the radius of convergence R. For $|z| < R$ we sum the series using common tricks for power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$$

Info.

$$not\_given$$

Comment.

$$no\_comment$$