Complex variable

solved problems

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11:03 May 29, 2012

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Acknowledgement. The following problems were solved using my own procedure in a program Maple V, release 5. All possible errors are my faults.
1 Residue theorem problems

We will solve several problems using the following theorem:

**Theorem.** (Residue theorem) Suppose $U$ is a simply connected open subset of the complex plane, and $w_1, \ldots, w_n$ are finitely many points of $U$ and $f$ is a function which is defined and holomorphic on $U \setminus \{w_1, \ldots, w_n\}$. If $\varphi$ is a simply closed curve in $U$ containing the points $w_k$ in the interior, then

$$\oint_{\varphi} f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{res} \left( f, w_k \right).$$

The following rules can be used for residue counting:

**Theorem.** (Rule 1) If $f$ has a pole of order $k$ at the point $w$ then

$$\text{res} \left( f, w \right) = \frac{1}{(k-1)!} \lim_{z \to w} \left( (z-w)^k f(z) \right)^{(k-1)}.$$

**Theorem.** (Rule 2) If $f, g$ are holomorphic at the point $w$ and $f(w) \neq 0$. If $g(w) = 0, g'(w) \neq 0$, then

$$\text{res} \left( \frac{f}{g}, w \right) = \frac{f(w)}{g'(w)}.$$

**Theorem.** (Rule 3) If $h$ is holomorphic at $w$ and $g$ has a simple pole at $w$, then

$$\text{res} \left( gh, w \right) = h(w) \text{res} \left( g, w \right).$$
We will use special formulas for special types of problems:

**Theorem.** (TYPE I. Integral from a rational function in sin and cos.) If $Q(a, b)$ is a rational function of two complex variables such that for real $a, b$, $a^2 + b^2 = 1$ is $Q(a, b)$ finite, then the function

$$T(z) := Q\left(\frac{z + 1/z}{2}, \frac{z - 1/z}{2i}\right)/(iz)$$

is rational, has no poles on the real line and

$$\int_0^{2\pi} Q(\cos t, \sin t) \, dt = 2\pi i \cdot \sum_{|a|<1, R(a)=\infty} \text{res} (T, a).$$

**Example**

$$\int_0^{2\pi} \frac{1}{2 + \cos(x)}\, dx$$
**Theorem.** (TYPE II. Integral from a rational function.) Suppose $P, Q$ are polynomial of order $m, n$ respectively, and $n - m > 1$ and $Q$ has no real roots. Then for the rational function $f = \frac{P}{Q}$ holds

$$
\int_{-\infty}^{+\infty} f(x) \, dx = 2\pi i \sum_{k} \text{res}(f, w_k),
$$

where all singularities of $f$ with a positive imaginary part are considered in the above sum.

**Example**

$$
\int_{-\infty}^{+\infty} \frac{1}{1 + x^2} \, dx
$$
Theorem. (TYPE III. Integral from a rational function multiplied by \( \cos \) or \( \sin \)) If \( Q \) is a rational function such that has no pole at the real line and for \( z \to \infty \) is \( Q(z) = O(z^{-1}) \). For \( b > 0 \) denote \( f(z) = Q(z) e^{ibz} \). Then

\[
\int_{-\infty}^{+\infty} Q(x) \cos(bx) \, dx = \Re \left( 2\pi i \cdot \sum_w \text{res}(f, w) \right)
\]

\[
\int_{-\infty}^{+\infty} Q(x) \sin(bx) \, dx = \Im \left( 2\pi i \cdot \sum_w \text{res}(f, w) \right)
\]

where only \( w \) with a positive imaginary part are considered in the above sums.

Example

\[
\int_{-\infty}^{+\infty} \frac{\cos(x)}{1 + x^2} \, dx
\]
1.1 Problem.

Using the Residue theorem evaluate
\[ \int_{0}^{2\pi} \frac{\cos(x)^2}{13 + 12 \cos(x)} \, dx \]

Hint.

Solution.

We denote
\[ f(z) = - \frac{I \left( \frac{1}{2} z + \frac{1}{2} \frac{1}{z} \right)^2}{(13 + 6 z + 6 \frac{1}{z}) z} \]

We find singularities
\[ \{ z = 0 \}, \{ z = -\frac{3}{2} \}, \{ z = -\frac{2}{3} \} \]

The singularity
\[ z = 0 \]

is in our region and we will add the following residue
\[ \text{res}(0, f(z)) = \frac{13}{144} I \]

The singularity
\[ z = -\frac{3}{2} \]

will be skipped because the singularity is not in our region.

The singularity
\[ z = -\frac{2}{3} \]

is in our region and we will add the following residue
\[ \text{res}(\frac{-2}{3}, f(z)) = -\frac{169}{720} I \]

Our sum is
\[ 2 I \pi \left( \sum \text{res}(z, f(z)) \right) = \frac{13}{45} \pi \]

The solution is
\[ \int_{0}^{2\pi} \frac{\cos(x)^2}{13 + 12 \cos(x)} \, dx = \frac{13}{45} \pi \]
We can try to solve it using real calculus and obtain the result

\[ \int_0^{2\pi} \frac{\cos(x)^2}{13 + 12 \cos(x)} \, dx = \frac{13}{45} \pi \]

Info.

\textit{not\_given}

Comment.

\textit{no\_comment}
1.2 Problem.

Using the Residue theorem evaluate

\[ \int_0^{2\pi} \frac{\cos(x)^4}{1 + \sin(x)^2} \, dx \]

Hint.

\[ T_{ype\_I} \]

Solution.

We denote

\[ f(z) = -\frac{I \left( \frac{1}{2} z + \frac{1}{2} \frac{1}{z} \right)^4}{(1 - \frac{1}{4} (z - \frac{1}{z})^2) z} \]

We find singularities

\[ \{ z = 0 \}, \{ z = \sqrt{2} + 1 \}, \{ z = 1 - \sqrt{2} \}, \{ z = \sqrt{2} - 1 \}, \{ z = -1 - \sqrt{2} \}, \{ z = \infty \}, \{ z = -\infty \} \]

The singularity

\[ z = 0 \]

is in our region and we will add the following residue

\[ \text{res}(0, f(z)) = \frac{5}{2} I \]

The singularity

\[ z = \sqrt{2} + 1 \]

will be skipped because the singularity is not in our region.

The singularity

\[ z = 1 - \sqrt{2} \]

is in our region and we will add the following residue

\[ \text{res}(1 - \sqrt{2}, f(z)) = \frac{-768 I \sqrt{2} + 1088 I}{768 - 544 \sqrt{2}} \]

The singularity

\[ z = \sqrt{2} - 1 \]

is in our region and we will add the following residue

\[ \text{res}(\sqrt{2} - 1, f(z)) = \frac{-768 I \sqrt{2} + 1088 I}{768 - 544 \sqrt{2}} \]

The singularity

\[ z = -1 - \sqrt{2} \]
will be skipped because the singularity is not in our region. The singularity

\[ z = \infty \]

will be skipped because only residues at finite singularities are counted. The singularity

\[ z = -\infty \]

will be skipped because only residues at finite singularities are counted. Our sum is

\[
2 I \pi \left( \sum \text{res}(z, f(z)) \right) = 2 I \pi \left( \frac{5}{2} I + 2 \frac{-768 I \sqrt{2} + 1088 I}{768 - 544 \sqrt{2}} \right)
\]

The solution is

\[
\int_{0}^{2\pi} \frac{\cos(x)^4}{1 + \sin(x)^2} \, dx = 2 I \pi \left( \frac{5}{2} I + 2 \frac{-768 I \sqrt{2} + 1088 I}{768 - 544 \sqrt{2}} \right)
\]

We can try to solve it using real calculus and obtain the result

\[
\int_{0}^{2\pi} \frac{\cos(x)^4}{1 + \sin(x)^2} \, dx = \frac{\pi (4 \sqrt{2} - 5)}{(\sqrt{2} + 1)(\sqrt{2} - 1)}
\]

Info.

not\_given

Comment.

no\_comment
1.3 Problem.

Using the Residue theorem evaluate

\[ \int_0^{2\pi} \frac{\sin(x)^2}{\frac{5}{4} - \cos(x)} \, dx \]

Hint.

\( \text{Type I} \)

Solution.

We denote

\[ f(z) = \frac{1}{4} \frac{I (z - \frac{1}{z})^2}{\left(\frac{5}{4} - \frac{1}{2} z - \frac{1}{2} \frac{1}{z}\right) z} \]

We find singularities

\[ \{ z = 0 \}, \{ z = \frac{1}{2} \}, \{ z = 2 \} \]

The singularity

\[ z = 0 \]

is in our region and we will add the following residue

\[ \text{res}(0, f(z)) = -\frac{5}{4} I \]

The singularity

\[ z = \frac{1}{2} \]

is in our region and we will add the following residue

\[ \text{res}\left(\frac{1}{2}, f(z)\right) = \frac{3}{4} I \]

The singularity

\[ z = 2 \]

will be skipped because the singularity is not in our region.

Our sum is

\[ 2 \pi I \left( \sum \text{res}(z, f(z)) \right) = \pi \]
The solution is

$$\int_0^{2\pi} \frac{\sin(x)^2}{\frac{5}{4} - \cos(x)} \, dx = \pi$$

We can try to solve it using real calculus and obtain the result

$$\int_0^{2\pi} \frac{\sin(x)^2}{\frac{5}{4} - \cos(x)} \, dx = \pi$$

Info.

\textit{not\_given}

Comment.

\textit{no\_comment}
1.4 Problem.
Using the Residue theorem evaluate
\[ \int_0^{2\pi} \frac{1}{\sin(x)^2 + 4\cos(x)^2} \, dx \]

Hint.
\[ Type \, I \]

Solution.
We denote
\[ f(z) = -\frac{I}{(-\frac{1}{4}(z - \frac{1}{z})^2 + 4(\frac{1}{2}z + \frac{1}{2}\frac{1}{z})^2)z} \]

We find singularities
\[ \{ z = -\frac{1}{3}I\sqrt{3}, \frac{1}{3}I\sqrt{3}, \frac{1}{3}I\sqrt{3}, \frac{1}{3}I\sqrt{3} \} \]
The singularity
\[ z = -\frac{1}{3}I\sqrt{3} \]
is in our region and we will add the following residue
\[ \text{res}(\frac{1}{3}I\sqrt{3}, f(z)) = -\frac{1}{4}I \]
The singularity
\[ z = \frac{1}{3}I\sqrt{3} \]
is in our region and we will add the following residue
\[ \text{res}(\frac{1}{3}I\sqrt{3}, f(z)) = -\frac{1}{4}I \]
The singularity
\[ z = I\sqrt{3} \]
will be skipped because the singularity is not in our region.
The singularity
\[ z = -I\sqrt{3} \]
will be skipped because the singularity is not in our region.
Our sum is
\[ 2I\pi \left( \sum \text{res}(z, f(z)) \right) = \pi \]
The solution is
\[\int_0^{2\pi} \frac{1}{\sin^2(x) + 4\cos^2(x)} \, dx = \pi\]

We can try to solve it using real calculus and obtain the result
\[\int_0^{2\pi} \frac{1}{\sin^2(x) + 4\cos^2(x)} \, dx = \pi\]

Info.

\textit{not\_given}

Comment.

\textit{no\_comment}
1.5 Problem.

Using the Residue theorem evaluate

\[ \int_{0}^{2\pi} \frac{1}{13 + 12 \sin(x)} \, dx \]

Hint.

Type I

Solution.

We denote

\[ f(z) = -\frac{I}{(13 - 6 I (z - \frac{1}{z})) z} \]

We find singularities

\[ \{ z = -\frac{2}{3} I \}, \{ z = -\frac{3}{2} I \} \]

The singularity

\[ z = -\frac{2}{3} I \]

is in our region and we will add the following residue

\[ \text{res}\left( -\frac{2}{3} I, f(z) \right) = -\frac{1}{5} I \]

The singularity

\[ z = -\frac{3}{2} I \]

will be skipped because the singularity is not in our region.

Our sum is

\[ 2 I \pi \left( \sum \text{res}(z, f(z)) \right) = \frac{2}{5} \pi \]

The solution is

\[ \int_{0}^{2\pi} \frac{1}{13 + 12 \sin(x)} \, dx = \frac{2}{5} \pi \]

We can try to solve it using real calculus and obtain the result

\[ \int_{0}^{2\pi} \frac{1}{13 + 12 \sin(x)} \, dx = \frac{2}{5} \pi \]

Info.

*not_given*
Comment.

no_comment
1.6 Problem.

Using the Residue theorem evaluate

\[ \int_0^{2\pi} \frac{1}{2 + \cos(x)} \, dx \]

Hint.

Type 1

Solution.

We denote

\[ f(z) = -\frac{I}{(2 + \frac{1}{2} z + \frac{1}{2} \frac{1}{z}) z} \]

We find singularities

\[ \{z = -2 + \sqrt{3}, \{z = -2 - \sqrt{3}\} \] \]

The singularity

\[ z = -2 + \sqrt{3} \]

is in our region and we will add the following residue

\[ \text{res}(-2 + \sqrt{3}, f(z)) = -\frac{1}{3} I \sqrt{3} \]

The singularity

\[ z = -2 - \sqrt{3} \]

will be skipped because the singularity is not in our region.

Our sum is

\[ 2 I \pi \left( \sum \text{res}(z, f(z)) \right) = \frac{2}{3} \pi \sqrt{3} \]

The solution is

\[ \int_0^{2\pi} \frac{1}{2 + \cos(x)} \, dx = \frac{2}{3} \pi \sqrt{3} \]

We can try to solve it using real calculus and obtain the result

\[ \int_0^{2\pi} \frac{1}{2 + \cos(x)} \, dx = \frac{2}{3} \pi \sqrt{3} \]

Info.

not_given

Comment.

no_comment
1.7 Problem.

Using the Residue theorem evaluate

\[ \int_0^{2\pi} \frac{1}{(2 + \cos(x))^2} \, dx \]

Hint.

\( Type \cdot I \)

Solution.

We denote

\[ f(z) = -\frac{I}{(2 + \frac{1}{2} z + \frac{1}{2} \frac{1}{z})^2 z} \]

We find singularities

\[ \{ z = -2 + \sqrt{3} \}, \{ z = -2 - \sqrt{3} \} \]

The singularity \( z = -2 + \sqrt{3} \) is in our region and we will add the following residue

\[ \text{res}( -2 + \sqrt{3} , f(z) ) = -\frac{1}{3} I + \frac{1}{3} (-\frac{2}{3} I + \frac{1}{3} I \sqrt{3}) \sqrt{3} \]

The singularity \( z = -2 - \sqrt{3} \) will be skipped because the singularity is not in our region.

Our sum is

\[ 2 I \pi \left( \sum \text{res}(z, f(z)) \right) = 2 I \pi \left( -\frac{1}{3} I + \frac{1}{3} (-\frac{2}{3} I + \frac{1}{3} I \sqrt{3}) \sqrt{3} \right) \]

The solution is

\[ \int_0^{2\pi} \frac{1}{(2 + \cos(x))^2} \, dx = 2 I \pi \left( -\frac{1}{3} I + \frac{1}{3} (-\frac{2}{3} I + \frac{1}{3} I \sqrt{3}) \sqrt{3} \right) \]

We can try to solve it using real calculus and obtain the result

\[ \int_0^{2\pi} \frac{1}{(2 + \cos(x))^2} \, dx = \frac{4}{9} \pi \sqrt{3} \]

Info.

\( not\_given \)

Comment.

\( no\_comment \)
1.8 Problem.

Using the Residue theorem evaluate

\[ \int_0^{2\pi} \frac{1}{2 + \sin(x)} \, dx \]

Hint.

Type I

Solution.

We denote

\[ f(z) = -\frac{I}{(2 - \frac{1}{2} I (z - \frac{1}{z})) z} \]

We find singularities

\[ \{ z = -2 I + I \sqrt{3} \}, \{ z = -2 I - I \sqrt{3} \} \]

The singularity

\[ z = -2 I + I \sqrt{3} \]

is in our region and we will add the following residue

\[ \text{res}(-2 I + I \sqrt{3}, f(z)) = -\frac{1}{3} I \sqrt{3} \]

The singularity

\[ z = -2 I - I \sqrt{3} \]

will be skipped because the singularity is not in our region.

Our sum is

\[ 2 I \pi \left( \sum \text{res}(z, f(z)) \right) = \frac{2}{3} \pi \sqrt{3} \]

The solution is

\[ \int_0^{2\pi} \frac{1}{2 + \sin(x)} \, dx = \frac{2}{3} \pi \sqrt{3} \]

We can try to solve it using real calculus and obtain the result

\[ \int_0^{2\pi} \frac{1}{2 + \sin(x)} \, dx = \frac{2}{3} \pi \sqrt{3} \]

Info.

\textit{not\_given}

Comment.

\textit{no\_comment}
1.9 Problem.

Using the Residue theorem evaluate

\[ \int_{0}^{2\pi} \frac{1}{5 + 4 \cos(x)} \, dx \]

Hint.

Type I

Solution.

We denote

\[ f(z) = -\frac{I}{(5 + 2z + 2 \frac{1}{z})z} \]

We find singularities

\[ \{ z = -2 \}, \{ z = -\frac{1}{2} \} \]

The singularity

\[ z = -2 \]

will be skipped because the singularity is not in our region.

The singularity

\[ z = -\frac{1}{2} \]

is in our region and we will add the following residue

\[ \text{res}(\frac{-1}{2}, f(z)) = -\frac{1}{3} I \]

Our sum is

\[ 2I\pi \left( \sum \text{res}(z, f(z)) \right) = \frac{2}{3} \pi \]

The solution is

\[ \int_{0}^{2\pi} \frac{1}{5 + 4 \cos(x)} \, dx = \frac{2}{3} \pi \]

We can try to solve it using real calculus and obtain the result

\[ \int_{0}^{2\pi} \frac{1}{5 + 4 \cos(x)} \, dx = \frac{2}{3} \pi \]

Info.

not given

Comment.

no comment
1.10 Problem.
Using the Residue theorem evaluate
\[ \int_{0}^{2\pi} \frac{1}{5 + 4\sin(x)} \, dx \]

Hint.
Type I

Solution.
We denote
\[ f(z) = -\frac{I}{(5 - 2I(z - \frac{1}{2}))z} \]

We find singularities
\[ \{ z = -2I, \{ z = -\frac{1}{2}I \} \} \]

The singularity
\[ z = -2I \]

will be skipped because the singularity is not in our region.

The singularity
\[ z = -\frac{1}{2}I \]
is in our region and we will add the following residue
\[ \text{res}(\frac{1}{2}I, f(z)) = -\frac{1}{3}I \]

Our sum is
\[ 2I\pi \left( \sum \text{res}(z, f(z)) \right) = \frac{2}{3}\pi \]

The solution is
\[ \int_{0}^{2\pi} \frac{1}{5 + 4\sin(x)} \, dx = \frac{2}{3}\pi \]

We can try to solve it using real calculus and obtain the result
\[ \int_{0}^{2\pi} \frac{1}{5 + 4\sin(x)} \, dx = \frac{2}{3}\pi \]

Info.
not_given

Comment.
no_comment
1.11 Problem.

Using the Residue theorem evaluate

\[ \int_{0}^{2\pi} \frac{\cos(x)}{\frac{5}{4} - \cos(x)} \, dx \]

Hint.

Type I

Solution.

We denote

\[ f(z) = -\frac{I \left( \frac{1}{2} z + \frac{1}{2} \frac{1}{z} \right)}{\left( \frac{5}{4} - \frac{1}{2} z - \frac{1}{2} \frac{1}{z} \right) z} \]

We find singularities

\[ \{ z = 0 \}, \{ z = \frac{1}{2} \}, \{ z = 2 \} \]

The singularity

\[ z = 0 \]

is in our region and we will add the following residue

\[ \text{res}(0, f(z)) = I \]

The singularity

\[ z = \frac{1}{2} \]

is in our region and we will add the following residue

\[ \text{res}\left( \frac{1}{2}, f(z) \right) = -\frac{5}{3} I \]

The singularity

\[ z = 2 \]

will be skipped because the singularity is not in our region.

Our sum is

\[ 2 I \pi \left( \sum \text{res}(z, f(z)) \right) = \frac{4}{3} \pi \]

The solution is

\[ \int_{0}^{2\pi} \frac{\cos(x)}{\frac{5}{4} - \cos(x)} \, dx = \frac{4}{3} \pi \]
We can try to solve it using real calculus and obtain the result

\[ \int_{0}^{2\pi} \frac{\cos(x)}{\frac{5}{4} - \cos(x)} \, dx = \frac{4}{3} \pi \]

Info.

not_given

Comment.

no_comment
1.12 Problem.

Using the Residue theorem evaluate

\[\int_{-\infty}^{\infty} \frac{x^2 + 1}{x^4 + 1} \, dx\]

Hint.

Type II

Solution.

We denote

\[f(z) = \frac{z^2 + 1}{z^4 + 1}\]

We find singularities

\[\left\{
\begin{aligned}
z &= -\frac{1}{2} \sqrt{2} - \frac{1}{2} I \sqrt{2} \\
z &= \frac{1}{2} \sqrt{2} + \frac{1}{2} I \sqrt{2} \\
z &= -\frac{1}{2} \sqrt{2} + \frac{1}{2} I \sqrt{2} \\
z &= \frac{1}{2} \sqrt{2} - \frac{1}{2} I \sqrt{2}
\end{aligned}\right.\]

The singularity \(z = -\frac{1}{2} \sqrt{2} - \frac{1}{2} I \sqrt{2}\)

will be skipped because the singularity is not in our region.

The singularity \(z = \frac{1}{2} \sqrt{2} + \frac{1}{2} I \sqrt{2}\)

is in our region and we will add the following residue

\[\text{res}\left(\frac{1}{2} \sqrt{2} + \frac{1}{2} I \sqrt{2}, f(z)\right) = \frac{1 + I}{2I \sqrt{2} - 2 \sqrt{2}}\]

The singularity \(z = -\frac{1}{2} \sqrt{2} + \frac{1}{2} I \sqrt{2}\)

is in our region and we will add the following residue

\[\text{res}\left(-\frac{1}{2} \sqrt{2} + \frac{1}{2} I \sqrt{2}, f(z)\right) = \frac{1 - I}{2I \sqrt{2} + 2 \sqrt{2}}\]

The singularity \(z = \frac{1}{2} \sqrt{2} - \frac{1}{2} I \sqrt{2}\)

will be skipped because the singularity is not in our region.

Our sum is

\[2I \pi \left(\sum \text{res}(z, f(z))\right) = 2I \pi \left(\frac{1 + I}{2I \sqrt{2} - 2 \sqrt{2}} + \frac{1 - I}{2I \sqrt{2} + 2 \sqrt{2}}\right)\]
The solution is

\[ \int_{-\infty}^{\infty} \frac{x^2 + 1}{x^4 + 1} \, dx = 2 \pi I \left( \frac{1 + I}{2 I \sqrt{2} - 2 \sqrt{2}} + \frac{1 - I}{2 I \sqrt{2} + 2 \sqrt{2}} \right) \]

We can try to solve it using real calculus and obtain the result

\[ \int_{-\infty}^{\infty} \frac{x^2 + 1}{x^4 + 1} \, dx = \sqrt{2} \pi \]

Info.

*not_given*

Comment.

*no_comment*
1.13 Problem.

Using the Residue theorem evaluate

\[ \int_{-\infty}^{\infty} \frac{x^2 - 1}{(x^2 + 1)^2} \, dx \]

Hint.

Type II

Solution.

We denote

\[ f(z) = \frac{z^2 - 1}{(z^2 + 1)^2} \]

We find singularities

\[ \{ z = -i \}, \{ z = i \} \]

The singularity \( z = i \) will be skipped because the singularity is not in our region.

The singularity \( z = -i \) is in our region and we will add the following residue

\[ \text{res}(I, f(z)) = 0 \]

Our sum is

\[ 2I \pi \left( \sum \text{res}(z, f(z)) \right) = 0 \]

The solution is

\[ \int_{-\infty}^{\infty} \frac{x^2 - 1}{(x^2 + 1)^2} \, dx = 0 \]

We can try to solve it using real calculus and obtain the result

\[ \int_{-\infty}^{\infty} \frac{x^2 - 1}{(x^2 + 1)^2} \, dx = 0 \]

Info.

\textit{not given}

Comment.

\textit{no comment}
1.14 Problem.
Using the Residue theorem evaluate
\[ \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} \, dx \]

Hint.

Type II

Solution.
We denote
\[ f(z) = \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} \]
We find singularities
\[ \{ z = 3i \}, \{ z = -3i \}, \{ z = -i \}, \{ z = i \} \]
The singularity
\[ z = 3i \]
is in our region and we will add the following residue
\[ \text{res}(3i, f(z)) = \frac{1}{16} - \frac{7}{48} \pi \]
The singularity
\[ z = -3i \]
will be skipped because the singularity is not in our region.
The singularity
\[ z = -i \]
will be skipped because the singularity is not in our region.
The singularity
\[ z = i \]
is in our region and we will add the following residue
\[ \text{res}(I, f(z)) = -\frac{1}{16} - \frac{1}{16} \pi \]
Our sum is
\[ 2 \pi \left( \sum \text{res}(z, f(z)) \right) = \frac{5}{12} \pi \]
The solution is
\[ \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} \, dx = \frac{5}{12} \pi \]

We can try to solve it using real calculus and obtain the result
\[ \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} \, dx = \frac{5}{12} \pi \]

Info.

\textit{not} \_\textit{given}

Comment.

\textit{no} \_\textit{comment}
1.15 Problem.
Using the Residue theorem evaluate
\[ \int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)(x^2 + 4)^2} \, dx \]

Hint.

Type II

Solution.
We denote
\[ f(z) = \frac{1}{(z^2 + 1)(z^2 + 4)^2} \]
We find singularities
\[ \{ z = 2i \}, \{ z = -i \}, \{ z = i \}, \{ z = -2i \} \]
The singularity
\[ z = 2i \]
is in our region and we will add the following residue
\[ \text{res}(2i, f(z)) = \frac{11}{288} i \]
The singularity
\[ z = -i \]
will be skipped because the singularity is not in our region.
The singularity
\[ z = i \]
is in our region and we will add the following residue
\[ \text{res}(i, f(z)) = -\frac{1}{18} i \]
The singularity
\[ z = -2i \]
will be skipped because the singularity is not in our region.
Our sum is
\[ 2i \pi \left( \sum \text{res}(z, f(z)) \right) = \frac{5}{144} \pi \]
The solution is

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)(x^2 + 4)^2} \, dx = \frac{5}{144} \pi$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)(x^2 + 4)^2} \, dx = \frac{5}{144} \pi$$

Info.

*not given*

Comment.

*no_comment*
1.16 Problem.

Using the Residue theorem evaluate

\[ \int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)(x^2 + 9)} \, dx \]

Hint.

Type II

Solution.

We denote

\[ f(z) = \frac{1}{(z^2 + 1)(z^2 + 9)} \]

We find singularities

\[ \{ z = 3i \}, \{ z = -3i \}, \{ z = -i \}, \{ z = i \} \]

The singularity \( z = 3i \) is in our region and we will add the following residue

\[ \text{res}(3i, f(z)) = \frac{1}{48}i \]

The singularity \( z = -3i \) will be skipped because the singularity is not in our region.

The singularity \( z = -i \) will be skipped because the singularity is not in our region.

The singularity \( z = i \) is in our region and we will add the following residue

\[ \text{res}(i, f(z)) = -\frac{1}{16}i \]

Our sum is

\[ 2i \pi \left( \sum \text{res}(z, f(z)) \right) = \frac{1}{12} \pi \]
The solution is

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)(x^2 + 9)} \, dx = \frac{1}{12} \pi$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)(x^2 + 9)} \, dx = \frac{1}{12} \pi$$

Info.

*not_given*

Comment.

*no_comment*
1.17 Problem.

Using the Residue theorem evaluate

\[ \int_{-\infty}^{\infty} \frac{1}{(x-I)(x-2I)} \, dx \]

Hint.

Type II

Solution.

We denote

\[ f(z) = \frac{1}{(z-I)(z-2I)} \]

We find singularities

\[ \{ z = 2I \}, \{ z = I \} \]

The singularity

\[ z = 2I \]

is in our region and we will add the following residue

\[ \text{res}(2I, f(z)) = -I \]

The singularity

\[ z = I \]

is in our region and we will add the following residue

\[ \text{res}(I, f(z)) = I \]

Our sum is

\[ 2I \pi \left( \sum \text{res}(z, f(z)) \right) = 0 \]

The solution is

\[ \int_{-\infty}^{\infty} \frac{1}{(x-I)(x-2I)} \, dx = 0 \]

We can try to solve it using real calculus and obtain the result

\[ \int_{-\infty}^{\infty} \frac{1}{(x-I)(x-2I)} \, dx = 0 \]

Info.

not given

Comment.

no comment
1.18 Problem.

Using the Residue theorem evaluate

\[ \int_{-\infty}^{\infty} \frac{1}{(x - I)(x - 2I)(x + 3I)} \, dx \]

Hint.

Type II

Solution.

We denote

\[ f(z) = \frac{1}{(z - I)(z - 2I)(z + 3I)} \]

We find singularities

\[ \{ z = 2I \}, \{ z = -3I \}, \{ z = I \} \]

The singularity \( z = 2I \) is in our region and we will add the following residue

\[ \text{res}(2I, f(z)) = -\frac{1}{5} \]

The singularity \( z = -3I \) will be skipped because the singularity is not in our region.

The singularity \( z = I \) is in our region and we will add the following residue

\[ \text{res}(I, f(z)) = \frac{1}{4} \]

Our sum is

\[ 2I \pi \left( \sum \text{res}(z, f(z)) \right) = \frac{1}{10} I \pi \]

The solution is

\[ \int_{-\infty}^{\infty} \frac{1}{(x - I)(x - 2I)(x + 3I)} \, dx = \frac{1}{10} I \pi \]
We can try to solve it using real calculus and obtain the result

\[ \int_{-\infty}^{\infty} \frac{1}{(x - I)(x - 2I)(x + 3I)} \, dx = \frac{1}{10} \, I \pi \]

Info.

\textit{not\_given}

Comment.

\textit{no\_comment}
1.19 Problem.
Using the Residue theorem evaluate
\[ \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx \]

Hint.
Type II

Solution.
We denote
\[ f(z) = \frac{1}{z^2 + 1} \]
We find singularities
\[ \{ z = -i \}, \{ z = i \} \]
The singularity
\[ z = -i \]
will be skipped because the singularity is not in our region.
The singularity
\[ z = i \]
is in our region and we will add the following residue
\[ \text{res}(I, f(z)) = -\frac{1}{2} i \]
Our sum is
\[ 2 I \pi \left( \sum \text{res}(z, f(z)) \right) = \pi \]
The solution is
\[ \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx = \pi \]
We can try to solve it using real calculus and obtain the result
\[ \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx = \pi \]

Info.

not given

Comment.

no_comment
1.20 Problem.

Using the Residue theorem evaluate

\[ \int_{-\infty}^{\infty} \frac{1}{x^3 + 1} \, dx \]

Hint.

Type II

Solution.

We denote

\[ f(z) = \frac{1}{z^3 + 1} \]

We find singularities

\[ \{ z = -1 \}, \{ z = \frac{1}{2} - \frac{1}{2} I \sqrt{3} \}, \{ z = \frac{1}{2} + \frac{1}{2} I \sqrt{3} \} \]

The singularity

\[ z = -1 \]

is on the real line and we will add one half of the following residue

\[ \text{res}(-1, f(z)) = \frac{1}{3} \]

The singularity

\[ z = \frac{1}{2} - \frac{1}{2} I \sqrt{3} \]

will be skipped because the singularity is not in our region.

The singularity

\[ z = \frac{1}{2} + \frac{1}{2} I \sqrt{3} \]

is in our region and we will add the following residue

\[ \text{res}(\frac{1}{2} + \frac{1}{2} I \sqrt{3}, f(z)) = 2 \frac{1}{3 I \sqrt{3} - 3} \]

Our sum is

\[ 2I \pi \left( \sum \text{res}(z, f(z)) \right) = 2I \pi \left( \frac{1}{6} + 2 \frac{1}{3 I \sqrt{3} - 3} \right) \]

The solution is

\[ \int_{-\infty}^{\infty} \frac{1}{x^3 + 1} \, dx = 2I \pi \left( \frac{1}{6} + 2 \frac{1}{3 I \sqrt{3} - 3} \right) \]
We can try to solve it using real calculus and obtain the result

\[ \int_{-\infty}^{\infty} \frac{1}{x^3 + 1} \, dx = \int_{-\infty}^{\infty} \frac{1}{x^3 + 1} \, dx \]

Info.

\textit{not\_given}

Comment.

\textit{no\_comment}
1.21 Problem.
Using the Residue theorem evaluate
\[ \int_{-\infty}^{\infty} \frac{1}{x^6 + 1} \, dx \]

Hint.

*Type II*

Solution.
We denote
\[ f(z) = \frac{1}{z^6 + 1} \]

We find singularities
\[ \{ z = -I \}, \{ z = I \}, \{ z = \frac{1}{2} \sqrt{2 + 2I \sqrt{3}} \}, \{ z = -\frac{1}{2} \sqrt{2 + 2I \sqrt{3}} \}, \{ z = \frac{1}{2} \sqrt{2 - 2I \sqrt{3}} \}, \{ z = -\frac{1}{2} \sqrt{2 - 2I \sqrt{3}} \} \]

The singularity
\[ z = -I \]

will be skipped because the singularity is not in our region.

The singularity
\[ z = I \]

is in our region and we will add the following residue
\[ \text{res}(I, f(z)) = \frac{-1}{6} I \]

The singularity
\[ z = \frac{1}{2} \sqrt{2 + 2I \sqrt{3}} \]

is in our region and we will add the following residue
\[ \text{res}(\frac{1}{2} \sqrt{2 + 2I \sqrt{3}}, f(z)) = \frac{16}{3} \frac{1}{(2 + 2I \sqrt{3})^{5/2}} \]

The singularity
\[ z = -\frac{1}{2} \sqrt{2 + 2I \sqrt{3}} \]

will be skipped because the singularity is not in our region.

The singularity
\[ z = \frac{1}{2} \sqrt{2 - 2I \sqrt{3}} \]
will be skipped because the singularity is not in our region. The singularity

\[ z = -\frac{1}{2} \sqrt{2 - 2 I \sqrt{3}} \]

is in our region and we will add the following residue

\[ \text{res}\left(-\frac{1}{2} \sqrt{2 - 2 I \sqrt{3}}, f(z)\right) = -\frac{16}{3} \frac{1}{(2 - 2 I \sqrt{3})^{5/2}} \]

Our sum is

\[ 2 I \pi \sum \text{res}(z, f(z)) = 2 I \pi \left(-\frac{1}{6} I + \frac{16}{3} \frac{1}{(2 + 2 I \sqrt{3})^{5/2}} - \frac{16}{3} \frac{1}{(2 - 2 I \sqrt{3})^{5/2}} \right) \]

The solution is

\[ \int_{-\infty}^{\infty} \frac{1}{x^6 + 1} \, dx = 2 I \pi \left(-\frac{1}{6} I + \frac{16}{3} \frac{1}{(2 + 2 I \sqrt{3})^{5/2}} - \frac{16}{3} \frac{1}{(2 - 2 I \sqrt{3})^{5/2}} \right) \]

We can try to solve it using real calculus and obtain the result

\[ \int_{-\infty}^{\infty} \frac{1}{x^6 + 1} \, dx = \frac{2}{3} \pi \]

Info.

*not given*

Comment.

*no comment*
1.22 Problem.
Using the Residue theorem evaluate
\[ \int_{-\infty}^{\infty} \frac{x}{(x^2 + 4x + 13)^2} \, dx \]

Hint.
Type II

Solution.
We denote
\[ f(z) = \frac{z}{(z^2 + 4z + 13)^2} \]

We find singularities
\[ \{ z = -2 + 3i \}, \{ z = -2 - 3i \} \]
The singularity
\[ z = -2 + 3i \]
is in our region and we will add the following residue
\[ \text{res}(-2 + 3i, f(z)) = \frac{1}{54}i \]
The singularity
\[ z = -2 - 3i \]
will be skipped because the singularity is not in our region.
Our sum is
\[ 2i \pi \left( \sum \text{res}(z, f(z)) \right) = -\frac{1}{27} \pi \]
The solution is
\[ \int_{-\infty}^{\infty} \frac{x}{(x^2 + 4x + 13)^2} \, dx = -\frac{1}{27} \pi \]

We can try to solve it using real calculus and obtain the result
\[ \int_{-\infty}^{\infty} \frac{x}{(x^2 + 4x + 13)^2} \, dx = -\frac{1}{27} \pi \]

Info.
not given

Comment.
no_comment
1.23 Problem.

Using the Residue theorem evaluate

\[ \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)(x^2 + 9)} \, dx \]

Hint.

Type II

Solution.

We denote

\[ f(z) = \frac{z^2}{(z^2 + 1)(z^2 + 9)} \]

We find singularities

\[ \{ z = 3i \}, \{ z = -3i \}, \{ z = -i \}, \{ z = i \} \]

The singularity \( z = 3i \) is in our region and we will add the following residue

\[ \text{res}(3i, f(z)) = -\frac{3}{16}i \]

The singularity \( z = -3i \) will be skipped because the singularity is not in our region.

The singularity \( z = -i \) will be skipped because the singularity is not in our region.

The singularity \( z = i \) is in our region and we will add the following residue

\[ \text{res}(i, f(z)) = \frac{1}{16}i \]

Our sum is

\[ 2i\pi \left( \sum \text{res}(z, f(z)) \right) = \frac{1}{4} \pi \]
The solution is

\[ \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)(x^2 + 9)} \, dx = \frac{1}{4} \pi \]

We can try to solve it using real calculus and obtain the result

\[ \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)(x^2 + 9)} \, dx = \frac{1}{4} \pi \]

Info.

\textit{not\_given}

Comment.

\textit{no\_comment}
1.24 Problem.
Using the Residue theorem evaluate
\[ \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)^3} \, dx \]

Hint.

Type II

Solution.
We denote
\[ f(z) = \frac{z^2}{(z^2 + 1)^3} \]
We find singularities
\[ \{ z = -I \}, \{ z = I \} \]
The singularity
\[ z = -I \]
will be skipped because the singularity is not in our region.
The singularity
\[ z = I \]
is in our region and we will add the following residue
\[ \text{res}(I, f(z)) = -\frac{1}{16} I \]
Our sum is
\[ 2I \pi \left( \sum \text{res}(z, f(z)) \right) = \frac{1}{8} \pi \]
The solution is
\[ \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)^3} \, dx = \frac{1}{8} \pi \]
We can try to solve it using real calculus and obtain the result
\[ \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)^3} \, dx = \frac{1}{8} \pi \]

Info.

\textit{not given}

Comment.

\textit{no comment}
1.25 Problem.
Using the Residue theorem evaluate 
\[ \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 4)^2} dx \]

Hint. 
\textit{Type II}

Solution.
We denote 
\[ f(z) = \frac{z^2}{(z^2 + 4)^2} \]
We find singularities 
\[ \{ z = 2i \}, \{ z = -2i \} \]
The singularity 
\[ z = 2i \]
is in our region and we will add the following residue 
\[ \text{res}(2i, f(z)) = -\frac{1}{8} i \]
The singularity 
\[ z = -2i \]
will be skipped because the singularity is not in our region.
Our sum is 
\[ 2i\pi \left( \sum \text{res}(z, f(z)) \right) = \frac{1}{4} \pi \]
The solution is 
\[ \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 4)^2} dx = \frac{1}{4} \pi \]
We can try to solve it using real calculus and obtain the result 
\[ \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 4)^2} dx = \frac{1}{4} \pi \]

Info. 
\textit{not given}

Comment. 
\textit{no comment}
1.26 **Problem.**

Using the Residue theorem evaluate

\[ \int_{-\infty}^{\infty} \frac{(x^3 + 5x) e^{(i\pi)} x^4 + 10x^2 + 9}{x^4 + 10x^2 + 9} \, dx \]

**Hint.**

*Type III*

**Solution.**

We denote

\[ f(z) = \frac{(z^3 + 5z) e^{(i\pi)} z^4 + 10z^2 + 9}{z^4 + 10z^2 + 9} \]

We find singularities

\[ \{ z = 3i \}, \{ z = -3i \}, \{ z = -1 \}, \{ z = 1 \} \]

The singularity

\[ z = 3i \]

is in our region and we will add the following residue

\[ \text{res}(3i, f(z)) = \frac{1}{4} e^{(-3)} \]

The singularity

\[ z = -3i \]

will be skipped because the singularity is not in our region. The singularity

\[ z = -1 \]

will be skipped because the singularity is not in our region.

The singularity

\[ z = i \]

is in our region and we will add the following residue

\[ \text{res}(i, f(z)) = \frac{1}{4} e^{(-1)} \]

Our sum is

\[ 2i \pi \left( \sum \text{res}(z, f(z)) \right) = 2i \pi \left( \frac{1}{4} e^{(-3)} + \frac{1}{4} e^{(-1)} \right) \]

45
The solution is
\[ \int_{-\infty}^{\infty} \frac{(x^3 + 5x) e^{ix}}{x^4 + 10x^2 + 9} \, dx = 2I\pi \left( \frac{1}{4} e^{(-3)} + \frac{1}{4} e^{(-1)} \right) \]

We can try to solve it using real calculus and obtain the result
\[ \int_{-\infty}^{\infty} \frac{(x^3 + 5x) e^{ix}}{x^4 + 10x^2 + 9} \, dx = \frac{1}{2} I \pi \cosh(1) + \frac{1}{2} I \pi \cosh(3) - \frac{1}{2} I \pi \sinh(3) - \frac{1}{2} I \pi \sinh(1) \]

Info.
\[ \text{not\_given} \]

Comment.
\[ \text{no\_comment} \]
1.27 Problem.

Using the Residue theorem evaluate

\[
\int_{-\infty}^{\infty} \frac{e^{(Ix)} (x^2 - 1)}{x (x^2 + 1)} \, dx
\]

Hint.

Type III

Solution.

We denote

\[ f(z) = \frac{e^{(Iz)} (z^2 - 1)}{z (z^2 + 1)} \]

We find singularities

\[ \{ z = 0 \}, \{ z = -I \}, \{ z = I \} \]

The singularity

\[ z = 0 \]

is on the real line and we will add one half of the following residue

\[ \text{res}(0, f(z)) = -1 \]

The singularity

\[ z = -I \]

will be skipped because the singularity is not in our region.

The singularity

\[ z = I \]

is in our region and we will add the following residue

\[ \text{res}(I, f(z)) = e^{(-1)} \]

Our sum is

\[ 2 I \pi \left( \sum \text{res}(z, f(z)) \right) = 2 I \pi \left( -\frac{1}{2} + e^{(-1)} \right) \]

The solution is

\[ \int_{-\infty}^{\infty} \frac{e^{(Ix)} (x^2 - 1)}{x (x^2 + 1)} \, dx = 2 I \pi \left( -\frac{1}{2} + e^{(-1)} \right) \]

We can try to solve it using real calculus and obtain the result

\[ \int_{-\infty}^{\infty} \frac{e^{(Ix)} (x^2 - 1)}{x (x^2 + 1)} \, dx = \int_{-\infty}^{\infty} \frac{(\cos(x) + I \sin(x)) (x^2 - 1)}{x (x^2 + 1)} \, dx \]
Info.

*not given*

Comment.

*no comment*
1.28 Problem.

Using the Residue theorem evaluate
\[ \int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{(x^2 + 4)(x - 1)} \, dx \]

Hint.

Type III

Solution.

We denote
\[ f(z) = \frac{e^{(I\,z)}}{(z^2 + 4)(z - 1)} \]

We find singularities
\[ \{ z = 2I \}, \{ z = -2I \}, \{ z = 1 \} \]

The singularity
\[ z = 2I \]

is in our region and we will add the following residue
\[ \text{res}(2I, f(z)) = \left( -\frac{1}{10} + \frac{1}{20} I \right) e^{(-2)} \]

The singularity
\[ z = -2I \]

will be skipped because the singularity is not in our region.

The singularity
\[ z = 1 \]

is on the real line and we will add one half of the following residue
\[ \text{res}(1, f(z)) = \frac{1}{5} e^{I} \]

Our sum is
\[ 2I\pi \left( \sum \text{res}(z, f(z)) \right) = 2I\pi \left( \left( -\frac{1}{10} + \frac{1}{20} I \right) e^{(-2)} + \frac{1}{10} e^{I} \right) \]

The solution is
\[ \int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{(x^2 + 4)(x - 1)} \, dx = 2I\pi \left( \left( -\frac{1}{10} + \frac{1}{20} I \right) e^{(-2)} + \frac{1}{10} e^{I} \right) \]
We can try to solve it using real calculus and obtain the result

\[
\int_{-\infty}^{\infty} \frac{e^{(I x)}}{(x^2 + 4)(x - 1)} \, dx = \int_{-\infty}^{\infty} \frac{\cos(x) + I \sin(x)}{(x^2 + 4)(x - 1)} \, dx
\]

Info.

\textit{not\_given}

Comment.

\textit{no\_comment}
1.29 Problem.

Using the Residue theorem evaluate

\[ \int_{-\infty}^{\infty} \frac{e^{(I x)}}{x(x^2 + 1)} \, dx \]

Hint.

Type III

Solution.

We denote

\[ f(z) = \frac{e^{(I z)}}{z(z^2 + 1)} \]

We find singularities

\[ \{ z = 0 \}, \{ z = -I \}, \{ z = I \} \]

The singularity \( z = 0 \) is on the real line and we will add one half of the following residue

\[ \text{res}(0, f(z)) = 1 \]

The singularity \( z = -I \) will be skipped because the singularity is not in our region.

The singularity \( z = I \) is in our region and we will add the following residue

\[ \text{res}(I, f(z)) = -\frac{1}{2} e^{(-1)} \]

Our sum is

\[ 2I\pi \left( \sum \text{res}(z, f(z)) \right) = 2I\pi \left( \frac{1}{2} - \frac{1}{2} e^{(-1)} \right) \]

The solution is

\[ \int_{-\infty}^{\infty} \frac{e^{(I x)}}{x(x^2 + 1)} \, dx = 2I\pi \left( \frac{1}{2} - \frac{1}{2} e^{(-1)} \right) \]
We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{e^{(Ix)}}{x(x^2+1)} \, dx = \int_{-\infty}^{\infty} \frac{\cos(x)+I\sin(x)}{x(x^2+1)} \, dx$$

Info.

not\_given

Comment.

no\_comment
1.30 Problem.
Using the Residue theorem evaluate
\[
\int_{-\infty}^{\infty} \frac{e^{(I x)}}{x (x^2 + 9)^2} \, dx
\]

Hint.

Solution.
We denote
\[
f(z) = \frac{e^{(I z)}}{z (z^2 + 9)^2}
\]
We find singularities
\[
\{ z = 0 \}, \{ z = 3 I \}, \{ z = -3 I \}
\]
The singularity
\[
z = 0
\]
is on the real line and we will add one half of the following residue
\[
\text{res}(0, f(z)) = \frac{1}{81}
\]
The singularity
\[
z = 3 I
\]
is in our region and we will add the following residue
\[
\text{res}(3 I, f(z)) = -\frac{5}{324} e^{(-3)}
\]
The singularity
\[
z = -3 I
\]
will be skipped because the singularity is not in our region.
Our sum is
\[
2 I \pi \left( \sum \text{res}(z, f(z)) \right) = 2 I \pi \left( \frac{1}{162} - \frac{5}{324} e^{(-3)} \right)
\]
The solution is
\[
\int_{-\infty}^{\infty} \frac{e^{(I x)}}{x (x^2 + 9)^2} \, dx = 2 I \pi \left( \frac{1}{162} - \frac{5}{324} e^{(-3)} \right)
\]
We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{e^{(i)x}}{x(x^2 + 9)^2} dx = \int_{-\infty}^{\infty} \frac{\cos(x) + i \sin(x)}{x(x^2 + 9)^2} dx$$

Info.

*not_given*

Comment.

*no_comment*
1.31 Problem.

Using the Residue theorem evaluate

\[ \int_{-\infty}^{\infty} \frac{e^{(1x)}}{x(x^2 - 2x + 2)} \, dx \]

Hint.

Type III

Solution.

We denote

\[ f(z) = \frac{e^{(1z)}}{z(z^2 - 2z + 2)} \]

We find singularities

\[ \{z = 0\}, \{z = 1 + I\}, \{z = 1 - I\} \]

The singularity

\[ z = 0 \]

is on the real line and we will add one half of the following residue

\[ \text{res}(0, f(z)) = \frac{1}{2} \]

The singularity

\[ z = 1 + I \]

is in our region and we will add the following residue

\[ \text{res}(1 + I, f(z)) = (-\frac{1}{4} - \frac{1}{4} I) e^{I} e^{(-1)} \]

The singularity

\[ z = 1 - I \]

will be skipped because the singularity is not in our region.

Our sum is

\[ 2I\pi \left( \sum \text{res}(z, f(z)) \right) = 2I\pi \left( \frac{1}{4} + (-\frac{1}{4} - \frac{1}{4} I) e^{I} e^{(-1)} \right) \]

The solution is

\[ \int_{-\infty}^{\infty} \frac{e^{(1x)}}{x(x^2 - 2x + 2)} \, dx = 2I\pi \left( \frac{1}{4} + (-\frac{1}{4} - \frac{1}{4} I) e^{I} e^{(-1)} \right) \]

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We can try to solve it using real calculus and obtain the result

\[
\int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x\,(x^2 - 2\,x + 2)} \, dx = \int_{-\infty}^{\infty} \frac{\cos(x) + I\,\sin(x)}{x\,(x^2 - 2\,x + 2)} \, dx
\]

Info.

\textit{not\_given}

Comment.

\textit{no\_comment}
1.32 Problem.

Using the Residue theorem evaluate

\[ \int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^2 + 1} \, dx \]

Hint.

Type III

Solution.

We denote

\[ f(z) = \frac{e^{(I\,z)}}{z^2 + 1} \]

We find singularities

\[ \{ z = -I \}, \{ z = I \} \]

The singularity

\[ z = -I \]

will be skipped because the singularity is not in our region.

The singularity

\[ z = I \]

is in our region and we will add the following residue

\[ \text{res}(I, f(z)) = -\frac{1}{2} I e^{(-1)} \]

Our sum is

\[ 2 \, I \, \pi \left( \sum \text{res}(z, f(z)) \right) = \pi e^{(-1)} \]

The solution is

\[ \int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^2 + 1} \, dx = \pi e^{(-1)} \]

We can try to solve it using real calculus and obtain the result

\[ \int_{-\infty}^{\infty} \frac{e^{(I\,x)}}{x^2 + 1} \, dx = -\pi \sinh(1) + \pi \cosh(1) \]

Info.

not given

Comment.

no comment
1.33 Problem.

Using the Residue theorem evaluate
\[ \int_{-\infty}^{\infty} \frac{e^{(I x)}}{x^2 + 4 x + 20} \, dx \]

Hint.

Type III

Solution.

We denote
\[ f(z) = \frac{e^{(I z)}}{z^2 + 4z + 20} \]

We find singularities
\[ \{z = -2 - 4I\}, \{z = -2 + 4I\} \]

The singularity
\[ z = -2 - 4I \]

will be skipped because the singularity is not in our region.

The singularity
\[ z = -2 + 4I \]

is in our region and we will add the following residue
\[ \text{res}(-2 + 4I, f(z)) = \frac{1}{8} \frac{I e^{(-4)}}{(eI)^2} \]

Our sum is
\[ 2I\pi \left( \sum \text{res}(z, f(z)) \right) = \frac{1}{4} \frac{\pi e^{(-4)}}{(eI)^2} \]

The solution is
\[ \int_{-\infty}^{\infty} \frac{e^{(I x)}}{x^2 + 4 x + 20} \, dx = \frac{1}{4} \frac{\pi}{(eI)^2} \]

We can try to solve it using real calculus and obtain the result
\[ \int_{-\infty}^{\infty} \frac{e^{(I x)}}{x^2 + 4 x + 20} \, dx = -\frac{1}{4} I\pi \sin(2 - 4I) + \frac{1}{4} \pi \cos(2 - 4I) \]

Info.

not given

Comment.

no_comment
1.34 Problem.

Using the Residue theorem evaluate
\[
\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 - 5x + 6} \, dx
\]

Hint.

Type III

Solution.

We denote
\[ f(z) = \frac{e^{iz}}{z^2 - 5z + 6} \]

We find singularities
\[ \{ z = 2 \}, \{ z = 3 \} \]

The singularity
\[ z = 2 \]

is on the real line and we will add one half of the following residue
\[ \text{res}(2, f(z)) = -(e^I)^2 \]

The singularity
\[ z = 3 \]

is on the real line and we will add one half of the following residue
\[ \text{res}(3, f(z)) = (e^I)^3 \]

Our sum is
\[
2 I \pi \left( \sum \text{res}(z, f(z)) \right) = 2 I \pi \left( -\frac{1}{2} (e^I)^2 + \frac{1}{2} (e^I)^3 \right)
\]

The solution is
\[
\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 - 5x + 6} \, dx = 2 I \pi \left( -\frac{1}{2} (e^I)^2 + \frac{1}{2} (e^I)^3 \right)
\]

We can try to solve it using real calculus and obtain the result
\[
\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 - 5x + 6} \, dx = \int_{-\infty}^{\infty} \frac{\cos(x) + I \sin(x)}{x^2 - 5x + 6} \, dx
\]

Info.

not given

Comment.

no comment
1.35 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{e^{(I x)}}{x^3 + 1} \, dx$$

Hint.

Type III

Solution.

We denote

$$f(z) = \frac{e^{(I z)}}{z^3 + 1}$$

We find singularities

$$\{z = -1\}, \{z = \frac{1}{2} - \frac{1}{2} I \sqrt{3}\}, \{z = \frac{1}{2} + \frac{1}{2} I \sqrt{3}\}$$

The singularity

$$z = -1$$

is on the real line and we will add one half of the following residue

$$\text{res}(-1, f(z)) = \frac{1}{3} e^{I}$$

The singularity

$$z = \frac{1}{2} - \frac{1}{2} I \sqrt{3}$$

will be skipped because the singularity is not in our region.

The singularity

$$z = \frac{1}{2} + \frac{1}{2} I \sqrt{3}$$

is in our region and we will add the following residue

$$\text{res}\left(\frac{1}{2} + \frac{1}{2} I \sqrt{3}, f(z)\right) = \frac{2}{3 I \sqrt{e^{(\sqrt{3})}} \sqrt{3} - 3 \sqrt{e^{(\sqrt{3})}}} (-1)^{(1/2 \pm \frac{1}{2})}$$

Our sum is

$$2 I \pi \left( \sum \text{res}(z, f(z)) \right) = 2 I \pi \left( \frac{1}{6} \frac{1}{e^{I}} + 2 \frac{(-1)^{(1/2 \pm \frac{1}{2})}}{3 I \sqrt{e^{(\sqrt{3})}} \sqrt{3} - 3 \sqrt{e^{(\sqrt{3})}}} \right)$$

The solution is

$$\int_{-\infty}^{\infty} \frac{e^{(I x)}}{x^3 + 1} \, dx = 2 I \pi \left( \frac{1}{6} \frac{1}{e^{I}} + 2 \frac{(-1)^{(1/2 \pm \frac{1}{2})}}{3 I \sqrt{e^{(\sqrt{3})}} \sqrt{3} - 3 \sqrt{e^{(\sqrt{3})}}} \right)$$

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We can try to solve it using real calculus and obtain the result:

\[ \int_{-\infty}^{\infty} \frac{e^{(I x)}}{x^3 + 1} \, dx = \int_{-\infty}^{\infty} \frac{\cos(x) + I \sin(x)}{x^3 + 1} \, dx \]

Info.

\textit{not\_given}

Comment.

\textit{no\_comment}
1.36 Problem.

Using the Residue theorem evaluate

\[ \int_{-\infty}^{\infty} \frac{e^{(I z)}}{x^4 - 1} \, dx \]

Hint.

Type III

Solution.

We denote

\[ f(z) = \frac{e^{(I z)}}{z^4 - 1} \]

We find singularities

\[ \{ z = -1 \}, \{ z = -I \}, \{ z = I \}, \{ z = 1 \} \]

The singularity

\[ z = -1 \]

is on the real line and we will add one half of the following residue

\[ \text{res}(-1, f(z)) = -\frac{1}{4} \frac{1}{e^I} \]

The singularity

\[ z = -I \]

will be skipped because the singularity is not in our region.

The singularity

\[ z = I \]

is in our region and we will add the following residue

\[ \text{res}(I, f(z)) = \frac{1}{4} I e^{(-1)} \]

The singularity

\[ z = 1 \]

is on the real line and we will add one half of the following residue

\[ \text{res}(1, f(z)) = \frac{1}{4} e^I \]

Our sum is

\[ 2 I \pi \left( \sum \text{res}(z, f(z)) \right) = 2 I \pi \left( -\frac{1}{8} \frac{1}{e^I} + \frac{1}{4} I e^{(-1)} + \frac{1}{8} e^I \right) \]

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The solution is

\[ \int_{-\infty}^{\infty} \frac{e^{ix}}{x^4 - 1} \, dx = 2 \pi \left( -\frac{1}{8} e^{i} + \frac{1}{4} I e^{(-1)} + \frac{1}{8} e^{i} \right) \]

We can try to solve it using real calculus and obtain the result

\[ \int_{-\infty}^{\infty} \frac{e^{ix}}{x^4 - 1} \, dx = \int_{-\infty}^{\infty} \frac{\cos(x) + I \sin(x)}{x^4 - 1} \, dx \]

Info.

not\_given

Comment.

no\_comment
1.37 Problem.
Using the Residue theorem evaluate
\[
\int_{-\infty}^{\infty} \frac{e^{(Ix)}}{x} \, dx
\]
Hint.
Type \text{ III}
Solution.
We denote
\[
f(z) = \frac{e^{(Iz)}}{z}
\]
We find singularities
\[
\{ z = 0 \}
\]
The singularity
\[
z = 0
\]
is on the real line and we will add one half of the following residue
\[
\text{res}(0, f(z)) = 1
\]
Our sum is
\[
2I\pi \left( \sum \text{res}(z, f(z)) \right) = I\pi
\]
The solution is
\[
\int_{-\infty}^{\infty} \frac{e^{(Ix)}}{x} \, dx = I\pi
\]
We can try to solve it using real calculus and obtain the result
\[
\int_{-\infty}^{\infty} \frac{e^{(Ix)}}{x} \, dx = \int_{-\infty}^{\infty} \frac{\cos(x) + I\sin(x)}{x} \, dx
\]
Info.
\text{ not_given }
Comment.
\text{ no_comment }
1.38 Problem.
Using the Residue theorem evaluate
\[ \int_{-\infty}^{\infty} \frac{x e^{(I x)}}{1 - x^4} \, dx \]

Hint.
*Type III*

Solution.
We denote
\[ f(z) = \frac{z e^{(I z)}}{1 - z^4} \]

We find singularities
\[ \{ z = -1, \ z = -I, \ z = I, \ z = 1 \} \]
The singularity
\[ z = -1 \]
is on the real line and we will add one half of the following residue
\[ \text{res}(-1, f(z)) = -\frac{1}{4} e^I \]
The singularity
\[ z = -I \]
will be skipped because the singularity is not in our region.
The singularity
\[ z = I \]
is in our region and we will add the following residue
\[ \text{res}(I, f(z)) = \frac{1}{4} e^{-1} \]
The singularity
\[ z = 1 \]
is on the real line and we will add one half of the following residue
\[ \text{res}(1, f(z)) = -\frac{1}{4} e^I \]

Our sum is
\[ 2 I \pi \left( \sum \text{res}(z, f(z)) \right) = 2 I \pi \left( -\frac{1}{8} e^I + \frac{1}{4} e^{-1} - \frac{1}{8} e^I \right) \]
The solution is
\[
\int_{-\infty}^{\infty} x e^{(Ix)} \frac{1}{1 - x^4} dx = 2 I \pi \left( -\frac{1}{8} e^{I} + \frac{1}{4} e^{(-1)} - \frac{1}{8} e^{I} \right)
\]
We can try to solve it using real calculus and obtain the result
\[
\int_{-\infty}^{\infty} x e^{(Ix)} \frac{1}{1 - x^4} dx = \int_{-\infty}^{\infty} -\frac{x (\cos(x) + I \sin(x))}{-1 + x^4} dx
\]
Info.

\texttt{not\_given}

Comment.

\texttt{no\_comment}
1.39 Problem.

Using the Residue theorem evaluate

\[ \int_{-\infty}^{\infty} \frac{x e^{(I x)}}{x^2 + 4x + 20} \, dx \]

Hint.

Type III

Solution.

We denote

\[ f(z) = \frac{z e^{(I z)}}{z^2 + 4z + 20} \]

We find singularities

\[ \{z = -2 - 4 I\}, \{z = -2 + 4 I\} \]

The singularity

\[ z = -2 - 4 I \]

will be skipped because the singularity is not in our region.

The singularity

\[ z = -2 + 4 I \]

is in our region and we will add the following residue

\[ \text{res}(-2 + 4 I, f(z)) = \frac{-1}{8} I (4 I e^{-4} - 2 e^{-4}) \]

Our sum is

\[ 2 I \pi \sum \text{res}(z, f(z)) = \frac{1}{4} \pi \left( 4 I e^{-4} - 2 e^{-4} \right) \]

The solution is

\[ \int_{-\infty}^{\infty} \frac{x e^{(I x)}}{x^2 + 4x + 20} \, dx = \frac{1}{4} \pi \left( 4 I e^{-4} - 2 e^{-4} \right) \]

We can try to solve it using real calculus and obtain the result

\[ \int_{-\infty}^{\infty} \frac{x e^{(I x)}}{x^2 + 4x + 20} \, dx = \pi \sin(2 - 4 I) + I \pi \cos(2 - 4 I) + \frac{1}{2} I \pi \sin(2 - 4 I) - \frac{1}{2} \pi \cos(2 - 4 I) \]
Info.

not_given

Comment.

no_comment
1.40 Problem.

Using the Residue theorem evaluate

\[ \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + 9} \, dx \]

Hint.

**Type III**

Solution.

We denote

\[ f(z) = \frac{ze^{iz}}{z^2 + 9} \]

We find singularities

\[ \{ z = 3i, z = -3i \} \]

The singularity \( z = 3i \) is in our region and we will add the following residue

\[ \text{res}(3i, f(z)) = \frac{1}{2} e^{-3} \]

The singularity \( z = -3i \) will be skipped because the singularity is not in our region.

Our sum is

\[ 2i \pi \left( \sum \text{res}(z, f(z)) \right) = i \pi e^{-3} \]

The solution is

\[ \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + 9} \, dx = i \pi e^{-3} \]

We can try to solve it using real calculus and obtain the result

\[ \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + 9} \, dx = i \pi \cosh(3) - i \pi \sinh(3) \]

Info.

*not given*

Comment.

*no_comment*
1.41 Problem.

Using the Residue theorem evaluate

\[
\int_{-\infty}^{\infty} \frac{x e^{(ix)}}{x^2 - 2x + 10} \, dx
\]

Hint.

Type III

Solution.

We denote

\[
f(z) = \frac{ze^{(iz)}}{z^2 - 2z + 10}
\]

We find singularities

\[
\{ z = 1 - 3I \}, \{ z = 1 + 3I \}
\]

The singularity 

\[
z = 1 - 3I
\]

will be skipped because the singularity is not in our region.

The singularity

\[
z = 1 + 3I
\]

is in our region and we will add the following residue

\[
\text{res}(1 + 3I, f(z)) = -\frac{1}{6} I(3I e^I e^{-3} + e^I e^{-3})
\]

Our sum is

\[
2I\pi \left( \sum \text{res}(z, f(z)) \right) = \frac{1}{3} \pi (3I e^I e^{-3} + e^I e^{-3})
\]

The solution is

\[
\int_{-\infty}^{\infty} \frac{x e^{(ix)}}{x^2 - 2x + 10} \, dx = \frac{1}{3} \pi (3I e^I e^{-3} + e^I e^{-3})
\]

We can try to solve it using real calculus and obtain the result

\[
\int_{-\infty}^{\infty} \frac{x e^{(ix)}}{x^2 - 2x + 10} \, dx =
\]

\[
-\pi \sin(1 + 3I) + I\pi \cos(1 + 3I) + \frac{1}{3} I\pi \sin(1 + 3I) + \frac{1}{3} \pi \cos(1 + 3I)
\]

Info.

\text{not\_given}

70
Comment.

no_comment
1.42 Problem.

Using the Residue theorem evaluate

$$\int_{-\infty}^{\infty} \frac{x e^{(I x)}}{x^2 - 5 x + 6} \, dx$$

Hint.

Type III

Solution.

We denote

$$f(z) = \frac{z e^{(I z)}}{z^2 - 5 z + 6}$$

We find singularities

$$\{ \{ z = 2 \}, \{ z = 3 \} \}$$

The singularity

$$z = 2$$

is on the real line and we will add one half of the following residue

$$\text{res}(2, f(z)) = -2 \,(e^I)^2$$

The singularity

$$z = 3$$

is on the real line and we will add one half of the following residue

$$\text{res}(3, f(z)) = 3 \,(e^I)^3$$

Our sum is

$$2 \, I \, \pi \left( \sum \text{res}(z, f(z)) \right) = 2 \, I \, \pi \left( - (e^I)^2 + \frac{3}{2} \,(e^I)^3 \right)$$

The solution is

$$\int_{-\infty}^{\infty} \frac{x e^{(I x)}}{x^2 - 5 x + 6} \, dx = 2 \, I \, \pi \left( - (e^I)^2 + \frac{3}{2} \,(e^I)^3 \right)$$

We can try to solve it using real calculus and obtain the result

$$\int_{-\infty}^{\infty} \frac{x e^{(I x)}}{x^2 - 5 x + 6} \, dx = \int_{-\infty}^{\infty} \frac{x (\cos(x) + I \sin(x))}{x^2 - 5 x + 6} \, dx$$

Info.

not_given

Comment.

no_comment
1.43 Problem.

Using the Residue theorem evaluate

\[ \int_{-\infty}^{\infty} e^{(I x)} \frac{x}{x^2} \, dx \]

Hint.

\[ Type \, III \]

Solution.

We denote

\[ f(z) = e^{(I z)} \frac{z}{z^2} \]

We find singularities

\[ \{ z = 0 \} \]

The singularity

\[ z = 0 \]

is on the real line and is not a simple pole, we cannot count the integral with the residue theorem ...

Our sum is

\[ 2 I \pi \left( \sum \text{res} (z, f(z)) \right) = 2 I \pi \infty \]

The solution is

\[ \int_{-\infty}^{\infty} e^{(I x)} \frac{x}{x^2} \, dx = 2 I \pi \infty \]

We can try to solve it using real calculus and obtain the result

\[ \int_{-\infty}^{\infty} e^{(I x)} \frac{x}{x^2} \, dx = \infty \]

Info.

\[ not\_given \]

Comment.

\[ no\_comment \]
1.44 Problem.

Using the Residue theorem evaluate

\[ \int_{-\infty}^{\infty} \frac{x^3 e^{(I x)}}{x^4 + 5x^2 + 4} \, dx \]

Hint.

Solution.

We denote

\[ f(z) = \frac{z^3 e^{(I z)}}{z^4 + 5z^2 + 4} \]

We find singularities

\[ \{ z = 2I, \{ z = -I \}, \{ z = I \}, \{ z = -2I \} \} \]

The singularity

\[ z = 2I \]

is in our region and we will add the following residue

\[ \text{res}(2I, f(z)) = \frac{2}{3} e^{(-2)} \]

The singularity

\[ z = -I \]

will be skipped because the singularity is not in our region.

The singularity

\[ z = I \]

is in our region and we will add the following residue

\[ \text{res}(I, f(z)) = -\frac{1}{6} e^{(-1)} \]

The singularity

\[ z = -2I \]

will be skipped because the singularity is not in our region.

Our sum is

\[ 2I \pi \left( \sum \text{res}(z, f(z)) \right) = 2I \pi \left( \frac{2}{3} e^{(-2)} - \frac{1}{6} e^{(-1)} \right) \]
The solution is
\[ \int_{-\infty}^{\infty} \frac{x^3 e(1 \cdot x)}{x^4 + 5x^2 + 4} \, dx = 2I\pi \left( \frac{2}{3} e(-2) - \frac{1}{6} e(-1) \right) \]

We can try to solve it using real calculus and obtain the result
\[ \int_{-\infty}^{\infty} \frac{x^3 e(1 \cdot x)}{x^4 + 5x^2 + 4} \, dx = -\frac{1}{3} I\pi \cosh(1) + \frac{4}{3} I\pi \cosh(2) - \frac{4}{3} I\pi \sinh(2) + \frac{1}{3} I\pi \sinh(1) \]

Info.

not_given

Comment.

no_comment
2 Zero Sum theorem for residues problems

We recall the definition:

**Notation.** For a function \( f \) holomorphic on some neighbourhood of infinity we define

\[
\text{res} (f, \infty) = \text{res} \left( \frac{-1}{z^2} \cdot f \left( \frac{1}{z} \right), 0 \right).
\]

We will solve several problems using the following theorem:

**Theorem.** (Zero Sum theorem for residues) For a function \( f \) holomorphic in the extended complex plane \( \mathbb{C} \cup \{ \infty \} \) with at most finitely many exceptions the sum of residues is zero, i.e.

\[
\sum_{w \in \mathbb{C} \cup \{ \infty \}} \text{res} (f, w) = 0.
\]

**Notation.** (Example \( f(z) = 1/z \)) Projecting on the Riemann sphere we observe the north pole on globe (i.e. \( \infty \)) with the residue -1 and at the south pole (the origin) with residue 1. Green is the zero level on the Riemann sphere while the blue gradually turning red is the imaginary part of \( \log z \) demonstrating that Zero Sum theorem holds for \( 1/z \).
2.1 Problem.

Check the Zero Sum theorem for the following function
\[ \frac{z^2 + 1}{z^4 + 1} \]

Hint.

no_hint

Solution.

We denote
\[ f(z) = \frac{z^2 + 1}{z^4 + 1} \]

We find singularities
\[ \left\{ z = \frac{1}{2}\sqrt{2} + \frac{1}{2} I \sqrt{2}, \right\} \]
\[ \left\{ z = -\frac{1}{2}\sqrt{2} - \frac{1}{2} I \sqrt{2}, \right\} \]
\[ \left\{ z = \frac{1}{2}\sqrt{2} - \frac{1}{2} I \sqrt{2}, \right\} \]
\[ \left\{ z = -\frac{1}{2}\sqrt{2} + \frac{1}{2} I \sqrt{2}, \right\} \]

The singularity
\[ z = \frac{1}{2}\sqrt{2} + \frac{1}{2} I \sqrt{2} \]

adds the following residue
\[ \text{res}(f(z), \frac{1}{2}\sqrt{2} + \frac{1}{2} I \sqrt{2}) = \frac{1 + I}{2 I \sqrt{2} - 2 \sqrt{2}} \]

The singularity
\[ z = -\frac{1}{2}\sqrt{2} - \frac{1}{2} I \sqrt{2} \]

adds the following residue
\[ \text{res}(f(z), -\frac{1}{2}\sqrt{2} - \frac{1}{2} I \sqrt{2}) = \frac{-1 - I}{2 I \sqrt{2} - 2 \sqrt{2}} \]

The singularity
\[ z = \frac{1}{2}\sqrt{2} - \frac{1}{2} I \sqrt{2} \]

adds the following residue
\[ \text{res}(f(z), \frac{1}{2}\sqrt{2} - \frac{1}{2} I \sqrt{2}) = \frac{-1 + I}{2 I \sqrt{2} + 2 \sqrt{2}} \]

The singularity
\[ z = -\frac{1}{2}\sqrt{2} + \frac{1}{2} I \sqrt{2} \]
adds the following residue
\[ \text{res}(f(z), -\frac{1}{2} \sqrt{2} + \frac{1}{2} I \sqrt{2}) = \frac{1 - I}{2 I \sqrt{2} + 2 \sqrt{2}} \]
At infinity we get the residue
\[ \text{res}(f(z), \infty) = \frac{1 + I}{2 I \sqrt{2} - 2 \sqrt{2}} + \frac{-1 - I}{2 I \sqrt{2} - 2 \sqrt{2}} + \frac{-1 + I}{2 I \sqrt{2} + 2 \sqrt{2}} + \frac{1 - I}{2 I \sqrt{2} + 2 \sqrt{2}} \]
and finally we obtain the sum
\[ \sum \text{res}(f(z), z) = 0 \]
Info.
\textit{not\_given}
Comment.
\textit{no\_comment}
2.2 Problem.

Check the Zero Sum theorem for the following function

\[ f(z) = \frac{z^2 - 1}{(z^2 + 1)^2} \]

Hint.

\textit{no\_hint}

Solution.

We denote

\[ f(z) = \frac{z^2 - 1}{(z^2 + 1)^2} \]

We find singularities

\[ \{z = -i\}, \{z = i\} \]

The singularity

\[ z = -i \]

adds the following residue

\[ \text{res}(f(z), -i) = 0 \]

The singularity

\[ z = i \]

adds the following residue

\[ \text{res}(f(z), i) = 0 \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = 0 \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info.

\textit{not\_given}

Comment.

\textit{no\_comment}
2.3 Problem.
Check the Zero Sum theorem for the following function
\[
\frac{z^2 - z + 2}{z^4 + 10z^2 + 9}
\]

Hint.
no_hint

Solution.
We denote
\[
f(z) = \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}
\]
We find singularities
\[
\left\{ z = -3i \right\}, \left\{ z = -i \right\}, \left\{ z = i \right\}, \left\{ z = 3i \right\}
\]
The singularity
\[
z = -3i
\]
adds the following residue
\[
\text{res}(f(z), -3i) = \frac{1}{16} + \frac{7}{48}i
\]
The singularity
\[
z = -i
\]
adds the following residue
\[
\text{res}(f(z), -i) = -\frac{1}{16} + \frac{1}{16}i
\]
The singularity
\[
z = i
\]
adds the following residue
\[
\text{res}(f(z), i) = -\frac{1}{16} - \frac{1}{16}i
\]
The singularity
\[
z = 3i
\]
adds the following residue
\[
\text{res}(f(z), 3i) = \frac{1}{16} - \frac{7}{48}i
\]
At infinity we get the residue
\[ \text{res}(f(z), \infty) = 0 \]
and finally we obtain the sum
\[ \sum \text{res}(f(z), z) = 0 \]

Info.
\[ \text{not\_given} \]

Comment.
\[ \text{no\_comment} \]
2.4 Problem.

Check the Zero Sum theorem for the following function

\[ f(z) = \frac{1}{(z^2 + 1)(z^2 + 4)^2} \]

Hint.

no_hint

Solution.

We denote

We find singularities

\[ \left\{ z = -I \right\}, \left\{ z = 2I \right\}, \left\{ z = -2I \right\}, \left\{ z = I \right\} \]

The singularity \( z = -I \) adds the following residue

\[ \text{res}(f(z), -I) = \frac{1}{18} I \]

The singularity \( z = 2I \) adds the following residue

\[ \text{res}(f(z), 2I) = \frac{11}{288} I \]

The singularity \( z = -2I \) adds the following residue

\[ \text{res}(f(z), -2I) = -\frac{11}{288} I \]

The singularity \( z = I \) adds the following residue

\[ \text{res}(f(z), I) = -\frac{1}{18} I \]
At infinity we get the residue
\[ \text{res}(f(z), \infty) = 0 \]
and finally we obtain the sum
\[ \sum \text{res}(f(z), z) = 0 \]

Info.
\[ \textit{not\_given} \]

Comment.
\[ \textit{no\_comment} \]
2.5 Problem.

Check the Zero Sum theorem for the following function

\[
\frac{1}{(z^2 + 1)(z^2 + 9)}
\]

Hint.

\[\text{no\_hint}\]

Solution.

We denote

\[f(z) = \frac{1}{(z^2 + 1)(z^2 + 9)}\]

We find singularities

\[\{z = -3\,i\}, \{z = -i\}, \{z = i\}, \{z = 3\,i\}\]

The singularity

\[z = -3\,i\]

adds the following residue

\[\text{res}(f(z), -3\,i) = -\frac{1}{48}\,i\]

The singularity

\[z = -i\]

adds the following residue

\[\text{res}(f(z), -i) = \frac{1}{16}\,i\]

The singularity

\[z = i\]

adds the following residue

\[\text{res}(f(z), i) = -\frac{1}{16}\,i\]

The singularity

\[z = 3\,i\]

adds the following residue

\[\text{res}(f(z), 3\,i) = \frac{1}{48}\,i\]
At infinity we get the residue
\[ \text{res}(f(z), \infty) = 0 \]
and finally we obtain the sum
\[ \sum \text{res}(f(z), z) = 0 \]

Info.

\text{not\_given}

Comment.

\text{no\_comment}
2.6 Problem.
Check the Zero Sum theorem for the following function

\[
\frac{1}{(z - i)(z - 2i)}
\]

Hint.

no Hint

Solution.
We denote

\[
f(z) = \frac{1}{(z - i)(z - 2i)}
\]

We find singularities

\[
\{ z = 2i \}, \{ z = i \}
\]

The singularity

\[
z = 2i
\]

adds the following residue

\[
\text{res}(f(z), 2i) = -i
\]

The singularity

\[
z = i
\]

adds the following residue

\[
\text{res}(f(z), i) = i
\]

At infinity we get the residue

\[
\text{res}(f(z), \infty) = 0
\]

and finally we obtain the sum

\[
\sum \text{res}(f(z), z) = 0
\]

Info.

not given

Comment.

no comment
2.7 Problem.

Check the Zero Sum theorem for the following function

\[
\frac{1}{(z - I)(z - 2I)(z + 3I)}
\]

Hint.

no_hint

Solution.

We denote

\[
f(z) = \frac{1}{(z - I)(z - 2I)(z + 3I)}
\]

We find singularities

\[\{z = -3I\}, \{z = 2I\}, \{z = I\}\]

The singularity

\[z = -3I\]

adds the following residue

\[\text{res}(f(z), -3I) = -\frac{1}{20}\]

The singularity

\[z = 2I\]

adds the following residue

\[\text{res}(f(z), 2I) = -\frac{1}{5}\]

The singularity

\[z = I\]

adds the following residue

\[\text{res}(f(z), I) = \frac{1}{4}\]

At infinity we get the residue

\[\text{res}(f(z), \infty) = 0\]

and finally we obtain the sum

\[\sum \text{res}(f(z), z) = 0\]

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Info.

not_given

Comment.

no_comment
2.8 Problem.

Check the Zero Sum theorem for the following function

\[ \frac{1}{z^2 + 1} \]

Hint.

no_hint

Solution.
We denote

\[ f(z) = \frac{1}{z^2 + 1} \]

We find singularities

\[ \{z = -i\}, \{z = i\} \]

The singularity

\[ z = -i \]

adds the following residue

\[ \text{res}(f(z), -i) = \frac{1}{2} i \]

The singularity

\[ z = i \]

adds the following residue

\[ \text{res}(f(z), i) = -\frac{1}{2} i \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = 0 \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info.

not_given

Comment.

no_comment

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2.9 Problem.
Check the Zero Sum theorem for the following function
\[ \frac{1}{z^3 + 1} \]

Hint.

no_hint

Solution.
We denote
\[ f(z) = \frac{1}{z^3 + 1} \]
We find singularities
\[ \{ z = -1 \}, \{ z = \frac{1}{2} - \frac{1}{2} I \sqrt{3} \}, \{ z = \frac{1}{2} + \frac{1}{2} I \sqrt{3} \} \]
The singularity
\[ z = -1 \]
adds the following residue
\[ \text{res}(f(z), -1) = \frac{1}{3} \]
The singularity
\[ z = \frac{1}{2} - \frac{1}{2} I \sqrt{3} \]
adds the following residue
\[ \text{res}(f(z), \frac{1}{2} - \frac{1}{2} I \sqrt{3}) = -2 \frac{1}{3 I \sqrt{3} + 3} \]
The singularity
\[ z = \frac{1}{2} + \frac{1}{2} I \sqrt{3} \]
adds the following residue
\[ \text{res}(f(z), \frac{1}{2} + \frac{1}{2} I \sqrt{3}) = 2 \frac{1}{3 I \sqrt{3} - 3} \]
At infinity we get the residue
\[ \text{res}(f(z), \infty) = -\frac{1}{3} + 2 \frac{1}{3 I \sqrt{3} + 3} - 2 \frac{1}{3 I \sqrt{3} - 3} \]
and finally we obtain the sum

$$\sum \operatorname{res}(f(z), z) = 0$$

Info.

\textit{not\_given}

Comment.

\textit{no\_comment}
2.10 Problem.

Check the Zero Sum theorem for the following function

\[ \frac{1}{z^6 + 1} \]

Hint.

no_hint

Solution.

We denote

\[ f(z) = \frac{1}{z^6 + 1} \]

We find singularities

\[ \{ z = -I \}, \{ z = -\frac{1}{2} \sqrt{2 - 2I \sqrt{3}} \}, \{ z = \frac{1}{2} \sqrt{2 - 2I \sqrt{3}} \}, \{ z = -\frac{1}{2} \sqrt{2 + 2I \sqrt{3}} \}, \{ z = \frac{1}{2} \sqrt{2 + 2I \sqrt{3}} \}, \{ z = I \} \]

The singularity

\[ z = -I \]

adds the following residue

\[ \text{res}(f(z), -I) = \frac{1}{6} I \]

The singularity

\[ z = -\frac{1}{2} \sqrt{2 - 2I \sqrt{3}} \]

adds the following residue

\[ \text{res}(f(z), -\frac{1}{2} \sqrt{2 - 2I \sqrt{3}}) = -\frac{16}{3} \frac{1}{(2 - 2I \sqrt{3})^{(5/2)}} \]

The singularity

\[ z = \frac{1}{2} \sqrt{2 - 2I \sqrt{3}} \]

adds the following residue

\[ \text{res}(f(z), \frac{1}{2} \sqrt{2 - 2I \sqrt{3}}) = \frac{16}{3} \frac{1}{(2 - 2I \sqrt{3})^{(5/2)}} \]
The singularity
\[ z = -\frac{1}{2} \sqrt{2 + 2I \sqrt{3}} \]
adds the following residue
\[ \text{res}(f(z), -\frac{1}{2} \sqrt{2 + 2I \sqrt{3}}) = -\frac{16}{3} \frac{1}{(2 + 2I \sqrt{3})^{5/2}} \]
The singularity
\[ z = \frac{1}{2} \sqrt{2 + 2I \sqrt{3}} \]
adds the following residue
\[ \text{res}(f(z), \frac{1}{2} \sqrt{2 + 2I \sqrt{3}}) = \frac{16}{3} \frac{1}{(2 + 2I \sqrt{3})^{5/2}} \]
The singularity
\[ z = I \]
adds the following residue
\[ \text{res}(f(z), I) = -\frac{1}{6} I \]
At infinity we get the residue
\[ \text{res}(f(z), \infty) = 0 \]
and finally we obtain the sum
\[ \sum \text{res}(f(z), z) = 0 \]

Info.
\[ \text{not\_given} \]

Comment.
\[ \text{no\_comment} \]
2.11 Problem.
Check the Zero Sum theorem for the following function

\[ \frac{z}{(z^2 + 4z + 13)^2} \]

Hint.
no_hint

Solution.
We denote

\[ f(z) = \frac{z}{(z^2 + 4z + 13)^2} \]

We find singularities

\[ \{ z = -2 - 3i \}, \{ z = -2 + 3i \} \]

The singularity

\[ z = -2 - 3i \]

adds the following residue

\[ \text{res}(f(z), -2 - 3i) = -\frac{1}{54}i \]

The singularity

\[ z = -2 + 3i \]

adds the following residue

\[ \text{res}(f(z), -2 + 3i) = \frac{1}{54}i \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = 0 \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info.
not_given

Comment.
no_comment
2.12 Problem.

Check the Zero Sum theorem for the following function

\[ f(z) = \frac{z^2}{(z^2 + 1)(z^2 + 9)} \]

Solution.

We denote

\[ f(z) = \frac{z^2}{(z^2 + 1)(z^2 + 9)} \]

We find singularities

\[ \{ z = -3i \}, \{ z = -i \}, \{ z = i \}, \{ z = 3i \} \]

The singularity

\[ z = -3i \]

adds the following residue

\[ \text{res}(f(z), -3i) = \frac{3}{16}i \]

The singularity

\[ z = -i \]

adds the following residue

\[ \text{res}(f(z), -i) = -\frac{1}{16}i \]

The singularity

\[ z = i \]

adds the following residue

\[ \text{res}(f(z), i) = \frac{1}{16}i \]

The singularity

\[ z = 3i \]

adds the following residue

\[ \text{res}(f(z), 3i) = -\frac{3}{16}i \]
At infinity we get the residue
\[ \text{res}(f(z), \infty) = 0 \]
and finally we obtain the sum
\[ \sum \text{res}(f(z), z) = 0 \]

Info.
\textit{not\_given}

Comment.
\textit{no\_comment}
2.13 Problem.

Check the Zero Sum theorem for the following function

\[ \frac{z^2}{(z^2 + 1)^3} \]

Hint.

no\_hint

Solution.

We denote

\[ f(z) = \frac{z^2}{(z^2 + 1)^3} \]

We find singularities

\[ \{ z = -1 \}, \{ z = 1 \} \]

The singularity

\[ z = -1 \]

adds the following residue

\[ \text{res}(f(z), -1) = \frac{1}{16} i \]

The singularity

\[ z = 1 \]

adds the following residue

\[ \text{res}(f(z), 1) = -\frac{1}{16} i \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = 0 \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info.

not\_given

Comment.

no\_comment

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2.14 Problem.

Check the Zero Sum theorem for the following function

\[
\frac{z^2}{(z^2 + 4)^2}
\]

Hint.

no_hint

Solution.

We denote

\[
f(z) = \frac{z^2}{(z^2 + 4)^2}
\]

We find singularities

\[
\{(z = 2I), (z = -2I)\}
\]

The singularity

\[z = 2I\]

adds the following residue

\[\text{res}(f(z), 2I) = -\frac{1}{8}I\]

The singularity

\[z = -2I\]

adds the following residue

\[\text{res}(f(z), -2I) = \frac{1}{8}I\]

At infinity we get the residue

\[\text{res}(f(z), \infty) = 0\]

and finally we obtain the sum

\[\sum \text{res}(f(z), z) = 0\]

Info.

not_given

Comment.

no_comment
2.15 Problem.

Check the Zero Sum theorem for the following function

\[
\frac{(z^3 + 5z)e^{(Iz)}}{z^4 + 10z^2 + 9}
\]

Hint.

no_hint

Solution.

We denote

\[
f(z) = \frac{(z^3 + 5z)e^{(Iz)}}{z^4 + 10z^2 + 9}
\]

We find singularities

\[
[z = -3I, \{z = -I\}, \{z = I\}, \{z = 3I\}]
\]

The singularity \(z = -3I\) adds the following residue

\[
\text{res}(f(z), -3I) = \frac{1}{4} e^3
\]

The singularity \(z = -I\) adds the following residue

\[
\text{res}(f(z), -I) = \frac{1}{4} e
\]

The singularity \(z = I\) adds the following residue

\[
\text{res}(f(z), I) = \frac{1}{4} e^{-3}
\]

The singularity \(z = 3I\) adds the following residue

\[
\text{res}(f(z), 3I) = \frac{1}{4} e^{-3}
\]
At infinity we get the residue

\[ \text{res}(f(z), \infty) = -\frac{1}{4} e^3 - \frac{1}{4} e - \frac{1}{4} e^{-1} - \frac{1}{4} e^{-3} \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info.

not given

Comment.

no comment
2.16 Problem.
Check the Zero Sum theorem for the following function

\[ e^{(Iz)} \frac{(z^2 - 1)}{z(z^2 + 1)} \]

Hint.

no_hint

Solution.
We denote

\[ f(z) = e^{(Iz)} \frac{(z^2 - 1)}{z(z^2 + 1)} \]

We find singularities

\[ \{ z = 0 \}, \{ z = -I \}, \{ z = I \} \]

The singularity \( z = 0 \)
add the following residue

\[ \text{res}(f(z), 0) = -1 \]

The singularity \( z = -I \)
add the following residue

\[ \text{res}(f(z), -I) = e \]

The singularity \( z = I \)
add the following residue

\[ \text{res}(f(z), I) = e^{(-1)} \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = 1 - e - e^{(-1)} \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]
Info.

not given

Comment.

no_comment
2.17 Problem.

Check the Zero Sum theorem for the following function

\[ \frac{e^{(Iz)}}{(z^2 + 4)(z - 1)} \]

Hint.

no_hint

Solution.

We denote

\[ f(z) = \frac{e^{(Iz)}}{(z^2 + 4)(z - 1)} \]

We find singularities

\[ \{ z = 1 \}, \{ z = 2I \}, \{ z = -2I \} \]

The singularity

\[ z = 1 \]

adds the following residue

\[ \text{res}(f(z), 1) = \frac{1}{5} e^I \]

The singularity

\[ z = 2I \]

adds the following residue

\[ \text{res}(f(z), 2I) = \left( -\frac{1}{10} + \frac{1}{20} I \right) e^{(-2)} \]

The singularity

\[ z = -2I \]

adds the following residue

\[ \text{res}(f(z), -2I) = \left( -\frac{1}{10} - \frac{1}{20} I \right) e^2 \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = -\frac{1}{5} e^I + \left( \frac{1}{10} - \frac{1}{20} I \right) e^{(-2)} + \left( \frac{1}{10} + \frac{1}{20} I \right) e^2 \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

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Info.

not_given

Comment.

no_comment
2.18 Problem.
Check the Zero Sum theorem for the following function

\[
\frac{e^{(Iz)}}{z(z^2 + 1)}
\]

Hint.
no hint

Solution.
We denote

\[
f(z) = \frac{e^{(Iz)}}{z(z^2 + 1)}
\]

We find singularities

\[
\{z = 0\}, \{z = -I\}, \{z = I\}
\]

The singularity \(z = 0\) adds the following residue

\[
\text{res}(f(z), 0) = 1
\]

The singularity \(z = -I\) adds the following residue

\[
\text{res}(f(z), -I) = -\frac{1}{2} e
\]

The singularity \(z = I\) adds the following residue

\[
\text{res}(f(z), I) = -\frac{1}{2} e^{(-1)}
\]

At infinity we get the residue

\[
\text{res}(f(z), \infty) = -1 + \frac{1}{2} e + \frac{1}{2} e^{(-1)}
\]

and finally we obtain the sum

\[
\sum \text{res}(f(z), z) = 0
\]
Info.

*not* given

Comment.

*no_comment*
2.19 Problem.

Check the Zero Sum theorem for the following function

\[ f(z) = \frac{e^{(I z)}}{z (z^2 + 9)^2} \]

Hint. 

no_hint

Solution.

We denote

\[ f(z) = \frac{e^{(I z)}}{z (z^2 + 9)^2} \]

We find singularities

\[ \{ \{ z = -3I \}, \{ z = 0 \}, \{ z = 3I \} \} \]

The singularity

\[ z = -3I \]

adds the following residue

\[ \text{res}(f(z), -3I) = \frac{1}{324} e^{3} \]

The singularity

\[ z = 0 \]

adds the following residue

\[ \text{res}(f(z), 0) = \frac{1}{81} \]

The singularity

\[ z = 3I \]

adds the following residue

\[ \text{res}(f(z), 3I) = -\frac{5}{324} e^{(-3)} \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = -\frac{1}{324} e^{3} - \frac{1}{81} + \frac{5}{324} e^{(-3)} \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]
Info.

not
given

Comment.

no
comment
2.20 Problem.
Check the Zero Sum theorem for the following function
\[
e^{(Iz)} \frac{z}{z^2 - 2z + 2}
\]

Hint.
no_hint

Solution.
We denote
\[
f(z) = \frac{e^{(Iz)}}{z(z^2 - 2z + 2)}
\]
We find singularities
\[
\{z = 0\}, \{z = 1 - I\}, \{z = 1 + I\}
\]
The singularity
\[
z = 0
\]
adds the following residue
\[
\text{res}(f(z), 0) = \frac{1}{2}
\]
The singularity
\[
z = 1 - I
\]
adds the following residue
\[
\text{res}(f(z), 1 - I) = \left(\frac{1}{4} + \frac{1}{4}I\right)e^I e
\]
The singularity
\[
z = 1 + I
\]
adds the following residue
\[
\text{res}(f(z), 1 + I) = \left(-\frac{1}{4} - \frac{1}{4}I\right)e^I e^{(-1)}
\]
At infinity we get the residue
\[
\text{res}(f(z), \infty) = \frac{1}{2} + \left(\frac{1}{4} - \frac{1}{4}I\right)e^I e + \left(\frac{1}{4} + \frac{1}{4}I\right)e^I e^{(-1)}
\]
and finally we obtain the sum
\[
\sum \text{res}(f(z), z) = 0
\]
Info.

not_given

Comment.

no_comment
2.21 Problem.

Check the Zero Sum theorem for the following function

\[ \frac{e^{(Iz)}}{z^2 + 1} \]

Hint.  

no_hint

Solution.

We denote

\[ f(z) = \frac{e^{(Iz)}}{z^2 + 1} \]

We find singularities

\[ \{(z = -I), \{z = I\}\} \]

The singularity  

\[ z = -I \]

adds the following residue

\[ \text{res}(f(z), -I) = \frac{1}{2} I e \]

The singularity  

\[ z = I \]

adds the following residue

\[ \text{res}(f(z), I) = -\frac{1}{2} I e^{(-1)} \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = -\frac{1}{2} I e + \frac{1}{2} I e^{(-1)} \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info.

not_given

Comment.

no_comment
2.22 Problem.

Check the Zero Sum theorem for the following function

\[ \frac{e^{(Iz)}}{z^2 + 4z + 20} \]

Hint.  
no_hint

Solution.

We denote

\[ f(z) = \frac{e^{(Iz)}}{z^2 + 4z + 20} \]

We find singularities

\[ \{ z = -2 - 4I \}, \{ z = -2 + 4I \} \]

The singularity

\[ z = -2 - 4I \]

adds the following residue

\[ \text{res}(f(z), -2 - 4I) = \frac{1}{8} \frac{Ie^4}{(e^I)^2} \]

The singularity

\[ z = -2 + 4I \]

adds the following residue

\[ \text{res}(f(z), -2 + 4I) = -\frac{1}{8} \frac{Ie^{(-4)}}{(e^I)^2} \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = -\frac{1}{8} \frac{Ie^4}{(e^I)^2} + \frac{1}{8} \frac{Ie^{(-4)}}{(e^I)^2} \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info.

not_given

Comment.

no_comment
2.23 Problem.

Check the Zero Sum theorem for the following function

\[
\frac{e^{(Iz)}}{z^2 - 5z + 6}
\]

Hint.

no_hint

Solution.

We denote

\[ f(z) = \frac{e^{(Iz)}}{z^2 - 5z + 6} \]

We find singularities

\[ \{z = 2\}, \{z = 3\} \]

The singularity \( z = 2 \) adds the following residue

\[ \text{res}(f(z), 2) = -e^I \]

The singularity \( z = 3 \) adds the following residue

\[ \text{res}(f(z), 3) = e^I \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = (e^I)^2 - (e^I)^3 \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info.

not_given

Comment.

no_comment
2.24 Problem.
Check the Zero Sum theorem for the following function
\[ e^{(Iz)} \]
\[ \frac{z^3 + 1}{3} \]

Hint.
\[ \text{no\_hint} \]

Solution.
We denote
\[ f(z) = \frac{e^{(Iz)}}{z^3 + 1} \]

We find singularities
\[ \{ z = -1 \}, \{ z = \frac{1}{2} - \frac{1}{2} I \sqrt{3} \}, \{ z = \frac{1}{2} + \frac{1}{2} I \sqrt{3} \} \]
The singularity
\[ z = -1 \]
adds the following residue
\[ \text{res}(f(z), -1) = \frac{1}{3} \frac{1}{e^I} \]
The singularity
\[ z = \frac{1}{2} - \frac{1}{2} I \sqrt{3} \]
adds the following residue
\[ \text{res}(f(z), \frac{1}{2} - \frac{1}{2} I \sqrt{3}) = -2 \frac{(-1)^{1/2} \sqrt{e(\sqrt{3})}}{3 I \sqrt{3} + 3} \]
The singularity
\[ z = \frac{1}{2} + \frac{1}{2} I \sqrt{3} \]
adds the following residue
\[ \text{res}(f(z), \frac{1}{2} + \frac{1}{2} I \sqrt{3}) = 2 \frac{(-1)^{1/2} \sqrt{e(\sqrt{3})}}{3 I \sqrt{e(\sqrt{3})} \sqrt{3} - 3 \sqrt{e(\sqrt{3})}} \]
At infinity we get the residue
\[ \text{res}(f(z), \infty) = -\frac{1}{3} \frac{1}{e^I} + 2 \frac{(-1)^{1/2} \sqrt{e(\sqrt{3})}}{3 I \sqrt{3} + 3} - 2 \frac{(-1)^{1/2}}{3 I \sqrt{e(\sqrt{3})} \sqrt{3} - 3 \sqrt{e(\sqrt{3})}} \]
and finally we obtain the sum
\[ \sum \text{res}(f(z), \ z) = 0 \]

Info.

*not_give*

Comment.

*no_comment*
2.25 **Problem.**

Check the Zero Sum theorem for the following function

\[ \frac{e^{(i \cdot z)}}{z^4 - 1} \]

**Hint.**

*no hint*

**Solution.**

We denote

\[ f(z) = \frac{e^{(i \cdot z)}}{z^4 - 1} \]

We find singularities

\[ \{ z = 1 \}, \{ z = -1 \}, \{ z = -I \}, \{ z = I \} \]

The singularity

\[ z = 1 \]

adds the following residue

\[ \text{res}(f(z), 1) = \frac{1}{4} e^I \]

The singularity

\[ z = -1 \]

adds the following residue

\[ \text{res}(f(z), -1) = -\frac{1}{4} \frac{1}{e^I} \]

The singularity

\[ z = -I \]

adds the following residue

\[ \text{res}(f(z), -I) = -\frac{1}{4} I e \]

The singularity

\[ z = I \]

adds the following residue

\[ \text{res}(f(z), I) = \frac{1}{4} I e^{(-1)} \]
At infinity we get the residue
\[
\text{res}(f(z), \infty) = -\frac{1}{4} e \ell + \frac{1}{4} \ell e + \frac{1}{4} \ell e - \frac{1}{4} \ell e^{(-1)}
\]
and finally we obtain the sum
\[
\sum \text{res}(f(z), z) = 0
\]
2.26 Problem.

Check the Zero Sum theorem for the following function

\[ f(z) = \frac{e^{(1z)}}{z} \]

Hint.

no_hint

Solution.

We denote

\[ f(z) = \frac{e^{(1z)}}{z} \]

We find singularities

\[ \{ z = 0 \} \]

The singularity

\[ z = 0 \]

adds the following residue

\[ \text{res}(f(z), 0) = 1 \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = -1 \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info.

not_given

Comment.

no_comment
2.27 Problem.

Check the Zero Sum theorem for the following function

\[ f(z) = \frac{ze^{(Iz)}}{1 - z^4} \]

Hint.

no hint

Solution.

We denote

\[ f(z) = \frac{ze^{(Iz)}}{1 - z^4} \]

We find singularities

\[ \{z = 1, \{z = -1, \{z = -I, \{z = I}\}\} \]

The singularity \( z = 1 \) adds the following residue

\[ \text{res}(f(z), 1) = -\frac{1}{4} e^I \]

The singularity \( z = -1 \) adds the following residue

\[ \text{res}(f(z), -1) = -\frac{1}{4} \frac{1}{e^I} \]

The singularity \( z = -I \) adds the following residue

\[ \text{res}(f(z), -I) = \frac{1}{4} e \]

The singularity \( z = I \) adds the following residue

\[ \text{res}(f(z), I) = \frac{1}{4} e^{(-1)} \]
At infinity we get the residue
\[ \text{res}(f(z), \infty) = \frac{1}{4} e^t + \frac{1}{4} e^{4t} - \frac{1}{4} e - \frac{1}{4} e^{(-1)} \]
and finally we obtain the sum
\[ \sum \text{res}(f(z), z) = 0 \]

Info.

\textit{not\_given}

Comment.

\textit{no\_comment}
2.28 Problem.

Check the Zero Sum theorem for the following function

\[ \frac{z e^{(Iz)}}{z^2 + 4z + 20} \]

Hint.

\textit{no\_hint}

Solution.

We denote

\[ f(z) = \frac{z e^{(Iz)}}{z^2 + 4z + 20} \]

We find singularities

\[ \{ z = -2 - 4I \}, \{ z = -2 + 4I \} \]

The singularity

\[ z = -2 - 4I \]

adds the following residue

\[ \text{res}(f(z), -2 - 4I) = -\frac{1}{8} \frac{I(4Ie^4 + 2e^4)}{(e^I)^2} \]

The singularity

\[ z = -2 + 4I \]

adds the following residue

\[ \text{res}(f(z), -2 + 4I) = -\frac{1}{8} \frac{I(4Ie^{-4} - 2e^{-4})}{(e^I)^2} \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = \frac{1}{8} \frac{I(4Ie^4 + 2e^4)}{(e^I)^2} + \frac{1}{8} \frac{I(4Ie^{-4} - 2e^{-4})}{(e^I)^2} \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info.

\textit{not\_given}

Comment.

\textit{no\_comment}
2.29 Problem.
Check the Zero Sum theorem for the following function
\[ \frac{ze^{(Iz)}}{z^2 + 9} \]

Hint.

no_hint

Solution.
We denote
\[ f(z) = \frac{ze^{(Iz)}}{z^2 + 9} \]
We find singularities
\[ \{ z = -3I \}, \{ z = 3I \} \]
The singularity
\[ z = -3I \]
adds the following residue
\[ \text{res}(f(z), -3I) = \frac{1}{2} e^3 \]
The singularity
\[ z = 3I \]
adds the following residue
\[ \text{res}(f(z), 3I) = \frac{1}{2} e^{(-3)} \]
At infinity we get the residue
\[ \text{res}(f(z), \infty) = -\frac{1}{2} e^3 - \frac{1}{2} e^{(-3)} \]
and finally we obtain the sum
\[ \sum \text{res}(f(z), z) = 0 \]

Info.

not_given

Comment.

no_comment
2.30 Problem.

Check the Zero Sum theorem for the following function

\[
\frac{z e^{(I z)}}{z^2 - 2 z + 10}
\]

Hint.

no_hint

Solution.

We denote

\[
f(z) = \frac{z e^{(I z)}}{z^2 - 2 z + 10}
\]

We find singularities

\[
\{z = 1 - 3 I\}, \{z = 1 + 3 I\}
\]

The singularity

\[
z = 1 - 3 I
\]

adds the following residue

\[
\text{res}(f(z), 1 - 3 I) = -\frac{1}{6} I (3 I e^I e^3 - e^I e^3)
\]

The singularity

\[
z = 1 + 3 I
\]

adds the following residue

\[
\text{res}(f(z), 1 + 3 I) = -\frac{1}{6} I (3 I e^I e^{-3} + e^I e^{-3})
\]

At infinity we get the residue

\[
\text{res}(f(z), \infty) = \frac{1}{6} I (3 I e^I e^3 - e^I e^3) + \frac{1}{6} I (3 I e^I e^{-3} + e^I e^{-3})
\]

and finally we obtain the sum

\[
\sum \text{res}(f(z), z) = 0
\]

Info.

not_given

Comment.

no_comment

123
2.31 Problem.

Check the Zero Sum theorem for the following function

\[
\frac{z e^{(Iz)}}{z^2 - 5z + 6}
\]

Hint.

no_hint

Solution.

We denote

\[
f(z) = \frac{z e^{(Iz)}}{z^2 - 5z + 6}
\]

We find singularities

[\{z = 2\}, \{z = 3\}]

The singularity

\[z = 2\]

adds the following residue

\[\text{res}(f(z), \ 2) = -2 \ (e^I)^2\]

The singularity

\[z = 3\]

adds the following residue

\[\text{res}(f(z), \ 3) = 3 \ (e^I)^3\]

At infinity we get the residue

\[\text{res}(f(z), \ \infty) = 2 \ (e^I)^2 - 3 \ (e^I)^3\]

and finally we obtain the sum

\[\sum \text{res}(f(z), \ z) = 0\]

Info.

not_given

Comment.

no_comment

124
2.32 Problem.

Check the Zero Sum theorem for the following function

\[ e^{(Iz)} \frac{z^2}{z^2} \]

Hint.

No hint

Solution.

We denote

\[ f(z) = e^{(Iz)} \frac{z^2}{z^2} \]

We find singularities

\[ \{ z = 0 \} \]

The singularity

\[ z = 0 \]

adds the following residue

\[ \text{res}(f(z), 0) = I \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = -I \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info.

Not given

Comment.

No comment
2.33 Problem.

Check the Zero Sum theorem for the following function

\[
\frac{z^3 e^{(Iz)}}{z^4 + 5z^2 + 4}
\]

Hint.

\textit{no\_hint}

Solution.

We denote

\[
f(z) = \frac{z^3 e^{(Iz)}}{z^4 + 5z^2 + 4}
\]

We find singularities

\[
\{z = -I, \{z = 2I\}, \{z = -2I\}, \{z = I\}\}
\]

The singularity

\[
z = -I
\]

adds the following residue

\[
\text{res}(f(z), -I) = -\frac{1}{6} e
\]

The singularity

\[
z = 2I
\]

adds the following residue

\[
\text{res}(f(z), 2I) = \frac{2}{3} e^{(-2)}
\]

The singularity

\[
z = -2I
\]

adds the following residue

\[
\text{res}(f(z), -2I) = \frac{2}{3} e^2
\]

The singularity

\[
z = I
\]

adds the following residue

\[
\text{res}(f(z), I) = -\frac{1}{6} e^{(-1)}
\]
At infinity we get the residue

\[ \text{res}(f(z), \infty) = \frac{1}{6} e - \frac{2}{3} e^{(-2)} - \frac{2}{3} e^2 + \frac{1}{6} e^{(-1)} \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info.

\textit{not\_given}

Comment.

\textit{no\_comment}
2.34 Problem.
Check the Zero Sum theorem for the following function
\[ f(z) = \frac{z^2 + 1}{e^z} \]

Hint.
no_hint

Solution.
We denote
\[ f(z) = \frac{z^2 + 1}{e^z} \]
We find singularities
\[ \{ z = -\infty \} \]
At infinity we get the residue
\[ \text{res}(f(z), \infty) = 0 \]
and finally we obtain the sum
\[ \sum \text{res}(f(z), z) = 0 \]

Info.
not_given

Comment.
no_comment
2.35 Problem.
Check the Zero Sum theorem for the following function

\[ \frac{z^2 + z - 1}{z^2 (z - 1)} \]

Hint. 

no_hint

Solution.
We denote

\[ f(z) = \frac{z^2 + z - 1}{z^2 (z - 1)} \]

We find singularities

\[ \{z = 1\}, \{z = 0\} \]

The singularity \( z = 1 \) adds the following residue

\[ \text{res}(f(z), 1) = 1 \]

The singularity \( z = 0 \) adds the following residue

\[ \text{res}(f(z), 0) = 0 \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = -1 \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info. 

not_given

Comment. 

no_comment
2.36 Problem.

Check the Zero Sum theorem for the following function

\[ f(z) = \frac{z^2 - 2z + 5}{(z - 2)(z^2 + 1)} \]

Hint.

no_hint

Solution.

We denote

\[ f(z) = \frac{z^2 - 2z + 5}{(z - 2)(z^2 + 1)} \]

We find singularities

\[ \{ z = 2 \}, \{ z = -i \}, \{ z = i \} \]

The singularity \( z = 2 \) adds the following residue

\[ \text{res}(f(z), 2) = 1 \]

The singularity \( z = -i \) adds the following residue

\[ \text{res}(f(z), -i) = -i \]

The singularity \( z = i \) adds the following residue

\[ \text{res}(f(z), i) = i \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = -1 \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]
Info.

not_given

Comment.

no_comment
2.37 Problem.
Check the Zero Sum theorem for the following function

\[ f(z) = \frac{1}{(z+1)(z-1)} \]

Hint.

no_hint

Solution.
We denote

\[ f(z) = \frac{1}{(z+1)(z-1)} \]

We find singularities

\[ \{z = 1\}, \{z = -1\} \]

The singularity \( z = 1 \) adds the following residue

\[ \text{res}(f(z), 1) = \frac{1}{2} \]

The singularity \( z = -1 \) adds the following residue

\[ \text{res}(f(z), -1) = -\frac{1}{2} \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = 0 \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info.

not_given

Comment.

no_comment

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2.38 **Problem.**

Check the Zero Sum theorem for the following function

\[ f(z) = \frac{1}{z(1-z^2)} \]

**Hint.**

no\_hint

**Solution.**

We denote

\[ f(z) = \frac{1}{z(1-z^2)} \]

We find singularities

\[ \{ z = 1 \}, \{ z = -1 \}, \{ z = 0 \} \]

The singularity

\[ z = 1 \]

adds the following residue

\[ \text{res}(f(z), 1) = \frac{-1}{2} \]

The singularity

\[ z = -1 \]

adds the following residue

\[ \text{res}(f(z), -1) = \frac{-1}{2} \]

The singularity

\[ z = 0 \]

adds the following residue

\[ \text{res}(f(z), 0) = 1 \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = 0 \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]
Info.

not_given

Comment.

no_comment
2.39 Problem.

Check the Zero Sum theorem for the following function

\[ f(z) = \frac{1}{z(z^2 + 4)^2} \]

Hint.

no_hint

Solution.

We denote

\[ f(z) = \frac{1}{z(z^2 + 4)^2} \]

We find singularities

\[ \{ z = 0 \}, \{ z = 2I \}, \{ z = -2I \} \]

The singularity

\[ z = 0 \]

adds the following residue

\[ \text{res}(f(z), 0) = \frac{1}{16} \]

The singularity

\[ z = 2I \]

adds the following residue

\[ \text{res}(f(z), 2I) = -\frac{1}{32} \]

The singularity

\[ z = -2I \]

adds the following residue

\[ \text{res}(f(z), -2I) = -\frac{1}{32} \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = 0 \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]
Info.

not_given

Comment.

no_comment
2.40 Problem.

Check the Zero Sum theorem for the following function

$$f(z) = \frac{1}{z(z-1)}$$

Hint.

no_hint

Solution.

We denote

$$f(z) = \frac{1}{z(z-1)}$$

We find singularities

$$\{z = 1, \{z = 0\}\}$$

The singularity

$$z = 1$$

adds the following residue

$$\text{res}(f(z), 1) = 1$$

The singularity

$$z = 0$$

adds the following residue

$$\text{res}(f(z), 0) = -1$$

At infinity we get the residue

$$\text{res}(f(z), \infty) = 0$$

and finally we obtain the sum

$$\sum \text{res}(f(z), z) = 0$$

Info.

not_given

Comment.

no_comment
2.41 Problem.

Check the Zero Sum theorem for the following function

\[ f(z) = \frac{1}{z(z - 1)} \]

Hint.

\textit{no hint}

Solution.

We denote

\[ f(z) = \frac{1}{z(z - 1)} \]

We find singularities \[ \{ z = 1, \ z = 0 \} \]

The singularity \[ z = 1 \]

adds the following residue \[ \text{res}(f(z), 1) = 1 \]

The singularity \[ z = 0 \]

adds the following residue \[ \text{res}(f(z), 0) = -1 \]

At infinity we get the residue \[ \text{res}(f(z), \infty) = 0 \]

and finally we obtain the sum \[ \sum \text{res}(f(z), z) = 0 \]

Info.

\textit{not given}

Comment.

\textit{no comment}
2.42 Problem.

Check the Zero Sum theorem for the following function

\[ f(z) = \frac{1}{(z^2 + 1)^2} \]

Hint.

no_hint

Solution.

We denote

\[ f(z) = \frac{1}{(z^2 + 1)^2} \]

We find singularities

\[ \{ z = -i \}, \{ z = i \} \]

The singularity

\[ z = -i \]

adds the following residue

\[ \text{res}(f(z), -i) = \frac{1}{4}i \]

The singularity

\[ z = i \]

adds the following residue

\[ \text{res}(f(z), i) = -\frac{1}{4}i \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = 0 \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info.

not_given

Comment.

no_comment

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2.43 Problem.

Check the Zero Sum theorem for the following function

\[
\frac{1}{z^3 - z^5}
\]

Hint.

no hint

Solution.

We denote

\[
f(z) = \frac{1}{z^3 - z^5}
\]

We find singularities

\[\{z = 1\}, \{z = -1\}, \{z = 0\}\]

The singularity

\[z = 1\]

adds the following residue

\[\text{res}(f(z), 1) = \frac{-1}{2}\]

The singularity

\[z = -1\]

adds the following residue

\[\text{res}(f(z), -1) = \frac{-1}{2}\]

The singularity

\[z = 0\]

adds the following residue

\[\text{res}(f(z), 0) = 1\]

At infinity we get the residue

\[\text{res}(f(z), \infty) = 0\]

and finally we obtain the sum

\[\sum \text{res}(f(z), z) = 0\]
Info. 

not_given

Comment.

no_comment
2.44 Problem.
Check the Zero Sum theorem for the following function

\[
\frac{1}{z^4 + 1}
\]

Hint.

no_hint

Solution.
We denote

\[ f(z) = \frac{1}{z^4 + 1} \]

We find singularities

\[
\{ 
\begin{align*}
  z &= \frac{1}{2} \sqrt{2} + \frac{1}{2} I \sqrt{2}, \\
  z &= -\frac{1}{2} \sqrt{2} - \frac{1}{2} I \sqrt{2}, \\
  z &= \frac{1}{2} \sqrt{2} - \frac{1}{2} I \sqrt{2}, \\
  z &= -\frac{1}{2} \sqrt{2} + \frac{1}{2} I \sqrt{2}
\end{align*}
\}
\]

The singularity

\[
 z = \frac{1}{2} \sqrt{2} + \frac{1}{2} I \sqrt{2}
\]
adds the following residue

\[
\text{res}(f(z), \frac{1}{2} \sqrt{2} + \frac{1}{2} I \sqrt{2}) = \frac{1}{2 I \sqrt{2} - 2 \sqrt{2}}
\]

The singularity

\[
 z = -\frac{1}{2} \sqrt{2} - \frac{1}{2} I \sqrt{2}
\]
adds the following residue

\[
\text{res}(f(z), -\frac{1}{2} \sqrt{2} - \frac{1}{2} I \sqrt{2}) = -\frac{1}{2 I \sqrt{2} - 2 \sqrt{2}}
\]

The singularity

\[
 z = \frac{1}{2} \sqrt{2} - \frac{1}{2} I \sqrt{2}
\]
adds the following residue

\[
\text{res}(f(z), \frac{1}{2} \sqrt{2} - \frac{1}{2} I \sqrt{2}) = -\frac{1}{2 I \sqrt{2} + 2 \sqrt{2}}
\]

The singularity

\[
 z = -\frac{1}{2} \sqrt{2} + \frac{1}{2} I \sqrt{2}
\]
adds the following residue
\[ \text{res}(f(z), -\frac{1}{2} \sqrt{2} + \frac{1}{2} I \sqrt{2}) = \frac{1}{2 I \sqrt{2} + 2 \sqrt{2}} \]
At infinity we get the residue
\[ \text{res}(f(z), \infty) = 0 \]
and finally we obtain the sum
\[ \sum \text{res}(f(z), z) = 0 \]

Info.

\textit{not\_given}

Comment.

\textit{no\_comment}
2.45 Problem.

Check the Zero Sum theorem for the following function

\[
\frac{1}{z - z^3}
\]

Hint.

no\_hint

Solution.

We denote

\[
f(z) = \frac{1}{z - z^3}
\]

We find singularities

\[
\{z = 1\}, \{z = -1\}, \{z = 0\}
\]

The singularity

\[
z = 1
\]

adds the following residue

\[
\text{res}(f(z), 1) = -\frac{1}{2}
\]

The singularity

\[
z = -1
\]

adds the following residue

\[
\text{res}(f(z), -1) = -\frac{1}{2}
\]

The singularity

\[
z = 0
\]

adds the following residue

\[
\text{res}(f(z), 0) = 1
\]

At infinity we get the residue

\[
\text{res}(f(z), \infty) = 0
\]

and finally we obtain the sum

\[
\sum \text{res}(f(z), z) = 0
\]
Info.

not_given

Comment.

no_comment
2.46 Problem.

Check the Zero Sum theorem for the following function

\[ f(z) = \frac{1}{z} \]

Hint. 

no_hint

Solution.

We denote

\[ f(z) = \frac{1}{z} \]

We find singularities

\[ \{ z = 0 \} \]

The singularity

\[ z = 0 \]

adds the following residue

\[ \text{res}(f(z), 0) = 1 \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = -1 \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info.

not_given

Comment.

no_comment
2.47 Problem.

Check the Zero Sum theorem for the following function

\[ \frac{e^z}{(z^2 + 9)z^2} \]

Hint.

no_hint

Solution.

We denote

\[ f(z) = \frac{e^z}{(z^2 + 9)z^2} \]

We find singularities

\[ \{ z = \infty \}, \{ z = -3i \}, \{ z = 0 \}, \{ z = 3i \} \]

The singularity

\[ z = -3i \]

adds the following residue

\[ \text{res}(f(z), -3i) = -\frac{1}{54} \frac{I}{(e^I)^3} \]

The singularity

\[ z = 0 \]

adds the following residue

\[ \text{res}(f(z), 0) = \frac{1}{9} \]

The singularity

\[ z = 3i \]

adds the following residue

\[ \text{res}(f(z), 3i) = \frac{1}{54} I (e^I)^3 \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = \frac{1}{54} \frac{I}{(e^I)^3} - \frac{1}{9} - \frac{1}{54} I (e^I)^3 \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]
Info.

not\_given

Comment.

no\_comment
2.48 Problem.

Check the Zero Sum theorem for the following function

\[ \frac{e^z}{z^2 + 1} \]

Hint.

no_hint

Solution.

We denote

\[ f(z) = \frac{e^z}{z^2 + 1} \]

We find singularities

\[ \{ z = \infty \}, \{ z = -I \}, \{ z = I \} \]

The singularity

\[ z = -I \]

adds the following residue

\[ \text{res}(f(z), -I) = \frac{1}{2} I e^I \]

The singularity

\[ z = I \]

adds the following residue

\[ \text{res}(f(z), I) = -\frac{1}{2} I e^I \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = -\frac{1}{2} I e^I + \frac{1}{2} I e^I \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info.

not_given

Comment.

no_comment
2.49 Problem.

Check the Zero Sum theorem for the following function

\[ f(z) = \frac{e^z}{(z^2 + 9) z^2} \]

Hint.

no\_hint

Solution.

We denote

\[ f(z) = \frac{e^z}{(z^2 + 9) z^2} \]

We find singularities

\[ \{ z = \infty \}, \{ z = -3I \}, \{ z = 0 \}, \{ z = 3I \} \]

The singularity

\[ z = -3I \]

adds the following residue

\[ \text{res}(f(z), -3I) = -\frac{1}{54} \frac{I}{(e^I)^3} \]

The singularity

\[ z = 0 \]

adds the following residue

\[ \text{res}(f(z), 0) = \frac{1}{9} \]

The singularity

\[ z = 3I \]

adds the following residue

\[ \text{res}(f(z), 3I) = \frac{1}{54} I (e^I)^3 \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = \frac{1}{54} \frac{I}{(e^I)^3} - \frac{1}{9} - \frac{1}{54} I (e^I)^3 \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]
Info.

not\_given

Comment.

no\_comment
2.50 Problem.
Check the Zero Sum theorem for the following function

\[
\frac{z}{(z-1)(z-2)^2}
\]

Hint.
no\_hint

Solution.
We denote

\[
f(z) = \frac{z}{(z-1)(z-2)^2}
\]

We find singularities

\[
\{z = 1\}, \{z = 2\}
\]

The singularity

\[
z = 1
\]

adds the following residue

\[
\text{res}(f(z), 1) = 1
\]

The singularity

\[
z = 2
\]

adds the following residue

\[
\text{res}(f(z), 2) = -1
\]

At infinity we get the residue

\[
\text{res}(f(z), \infty) = 0
\]

and finally we obtain the sum

\[
\sum \text{res}(f(z), z) = 0
\]

Info.
not\_given

Comment.
no\_comment
2.51 Problem.
Check the Zero Sum theorem for the following function

$$f(z) = \frac{z^2}{(z^2 + 1)^2}$$

Hint. 
no_hint

Solution.
We denote

$$f(z) = \frac{z^2}{(z^2 + 1)^2}$$

We find singularities

$$\{ \{ z = -i \}, \{ z = i \} \}$$

The singularity

$$z = -i$$

adds the following residue

$$\text{res}(f(z), -i) = \frac{1}{4} i$$

The singularity

$$z = i$$

adds the following residue

$$\text{res}(f(z), i) = -\frac{1}{4} i$$

At infinity we get the residue

$$\text{res}(f(z), \infty) = 0$$

and finally we obtain the sum

$$\sum \text{res}(f(z), z) = 0$$

Info.
not_given

Comment.
no_comment

153
2.52 Problem.

Check the Zero Sum theorem for the following function

\[ \frac{z^4}{z^4 + 1} \]

Hint.

no_hint

Solution.

We denote

\[ f(z) = \frac{z^4}{z^4 + 1} \]

We find singularities

\[ \{ z = \frac{1}{2} \sqrt{2} + \frac{1}{2} i \sqrt{2} \}, \{ z = -\frac{1}{2} \sqrt{2} - \frac{1}{2} i \sqrt{2} \}, \{ z = \frac{1}{2} \sqrt{2} - \frac{1}{2} i \sqrt{2} \}, \{ z = -\frac{1}{2} \sqrt{2} - \frac{1}{2} i \sqrt{2} \} \]

The singularity

\[ z = \frac{1}{2} \sqrt{2} + \frac{1}{2} i \sqrt{2} \]

adds the following residue

\[ \text{res}(f(z), \left( \frac{1}{2} \sqrt{2} + \frac{1}{2} i \sqrt{2} \right)) = -\frac{1}{2 i \sqrt{2} - 2 \sqrt{2}} \]

The singularity

\[ z = -\frac{1}{2} \sqrt{2} - \frac{1}{2} i \sqrt{2} \]

adds the following residue

\[ \text{res}(f(z), \left( -\frac{1}{2} \sqrt{2} - \frac{1}{2} i \sqrt{2} \right)) = \frac{1}{2 i \sqrt{2} - 2 \sqrt{2}} \]

The singularity

\[ z = \frac{1}{2} \sqrt{2} - \frac{1}{2} i \sqrt{2} \]

adds the following residue

\[ \text{res}(f(z), \left( \frac{1}{2} \sqrt{2} - \frac{1}{2} i \sqrt{2} \right)) = \frac{1}{2 i \sqrt{2} + 2 \sqrt{2}} \]

The singularity

\[ z = -\frac{1}{2} \sqrt{2} + \frac{1}{2} i \sqrt{2} \]
adds the following residue

$$\text{res}(f(z), -\frac{1}{2} \sqrt{2} + \frac{1}{2} I \sqrt{2}) = -\frac{1}{2 I \sqrt{2} + 2 \sqrt{2}}$$

At infinity we get the residue

$$\text{res}(f(z), \infty) = 0$$

and finally we obtain the sum

$$\sum \text{res}(f(z), z) = 0$$

Info.

not\_given

Comment.

no\_comment
2.53 Problem.

Check the Zero Sum theorem for the following function

\[ \frac{z^5}{(1 - z)^2} \]

Hint.

no\_hint

Solution.

We denote

\[ f(z) = \frac{z^5}{(1 - z)^2} \]

We find singularities

\[ \{ z = \infty \}, \{ z = 1 \}, \{ z = -\infty \} \]

The singularity

\[ z = 1 \]

adds the following residue

\[ \text{res}(f(z), 1) = 5 \]

At infinity we get the residue

\[ \text{res}(f(z), \infty) = -5 \]

and finally we obtain the sum

\[ \sum \text{res}(f(z), z) = 0 \]

Info.

not\_given

Comment.

no\_comment
3 Power series problems

Example

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}$$
3.1 Problem.

Sum the following power series
\[ \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)} z^n}{n} \]

Hint.

\textit{derive\_once\_and\_sum}

Solution.

We denote the \(n\)-th term in the series by
\[ a(n) = \frac{(-1)^{(n+1)} z^n}{n} \]

For the ratio test we need the term \(a(n+1)\)
\[ a(n+1) = \frac{(-1)^{(n+2)} z^{(n+1)}}{n+1} \]

Ratio test computes
\[ \lim_{n \to \infty} \left| \frac{a(n+1)}{a(n)} \right| \]

and obtains in our case
\[ \lim_{n \to \infty} \left| \frac{z^{(n+1)} n}{(n + 1) z^n} \right| = |z| \]

Moreover we can check the root test computing
\[ \lim_{n \to \infty} \sqrt[n]{|a(n)|} \]

and obtain in our case
\[ \lim_{n \to \infty} \left[ \frac{(-1)^{(n+1)} z^n}{n} \right]^{\frac{1}{n}} = |z| \]

From this we conclude the radius of convergence \(R\). For \(|z| < R\) we sum the series using common tricks for power series
\[ \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)} z^n}{n} = \ln(1 + z) \]

Info.

\textit{not\_given}

Comment.

\textit{no\_comment}

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3.2 Problem.

Sum the following power series

$$
\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{z^n}
$$

Hint.

*divide by $z$ and integrate*

Solution.

We denote the $n$-th term in the series by

$$a(n) = (-1)^n \frac{n^2}{z^n}$$

For the ratio test we need the term $a(n+1)$

$$a(n+1) = (-1)^{(n+1)} \frac{(n+1)^2}{z^{n+1}}$$

Ratio test computes

$$\lim_{n \to \infty} \left| \frac{a(n+1)}{a(n)} \right|$$

and obtains in our case

$$\lim_{n \to \infty} \left| \frac{(n+1)^2}{n^2} \frac{z^{n+1}}{z^n} \right| = |z|$$

Moreover we can check the root test computing

$$\lim_{n \to \infty} \sqrt[n]{|a(n)|}$$

and obtain in our case

$$\lim_{n \to \infty} \left| (-1)^n \frac{n^2}{z^n} \right|^{\frac{1}{n}} = |z|$$

From this we conclude the radius of convergence $R$. For $|z| < R$ we sum the series using common tricks for power series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{z^n} = -\frac{z (-z + 1)}{(1 + z)^3}$$

Info.

*not_given*

Comment.

*no_comment*
3.3 Problem.

Sum the following power series

\[ \sum_{n=0}^{\infty} \frac{(-1)^n n^3 z^n}{(n+1)!} \]

Hint.

*manipulate the numerator to cancel*

Solution.

We denote the \( n \)-th term in the series by

\[ a(n) = \frac{(-1)^n n^3 z^n}{(n+1)!} \]

For the ratio test we need the term \( a(n+1) \)

\[ a(n+1) = \frac{(-1)^{(n+1)} (n+1)^3 z^{(n+1)}}{(n+2)!} \]

Ratio test computes

\[ \lim_{n \to \infty} \left| \frac{a(n+1)}{a(n)} \right| \]

and obtains in our case

\[ \lim_{n \to \infty} \left| \frac{(n+1)^3 z^{(n+1)} (n+1)!}{(n+2)! n^3 z^n} \right| = 0 \]

Moreover we can check the root test computing

\[ \lim_{n \to \infty} \sqrt[n]{|a(n)|} \]

and obtain in our case

\[ \lim_{n \to \infty} \left( \frac{(-1)^n n^3 z^n}{(n+1)!} \right)^{\frac{1}{n}} = \left| \lim_{n \to \infty} \left( \frac{(-1)^n n^3 z^n}{(n+1)!} \right)^{\frac{1}{n}} \right| \]

From this we conclude the radius of convergence \( R \). For \( |z| < R \) we sum the series using common tricks for power series

\[ \sum_{n=0}^{\infty} \frac{(-1)^n n^3 z^n}{(n+1)!} = \]

\[ -1 \frac{z}{2} \left( 2 \frac{1 - e^{-z}}{z} \frac{1 + z}{z^2} - 12 \frac{1 - e^{-z}}{z^2} \frac{1 + z + \frac{1}{5} z^2}{z^2} + 12 \frac{1 - e^{-z}}{z^2} \frac{1 + z + \frac{1}{2} z^2 + \frac{1}{6} z^3}{z^2} \right) \]
Info.
not_given

Comment.
no_comment
3.4 Problem.

Sum the following power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n(2n-1)}$$

Hint.

*make_the_power_2n_and_derive_twice*

Solution.

We denote the $n$-th term in the series by

$$a(n) = \frac{(-1)^n z^n}{n(2n-1)}$$

For the ratio test we need the term $a(n+1)$

$$a(n+1) = \frac{(-1)^{(n+1)} z^{(n+1)}}{(n+1)(2n+1)}$$

Ratio test computes

$$\lim_{n \to \infty} \left| \frac{a(n+1)}{a(n)} \right|$$

and obtains in our case

$$\lim_{n \to \infty} \left| \frac{z^{(n+1)} n (2n-1)}{(n+1)(2n+1) z^n} \right| = |z|$$

Moreover we can check the root test computing

$$\lim_{n \to \infty} \sqrt[n]{|a(n)|}$$

and obtain in our case

$$\lim_{n \to \infty} \left| \frac{(-1)^n z^n}{n(2n-1)} \right|^{(\frac{1}{n})} = |z|$$

From this we conclude the radius of convergence $R$. For $|z| < R$ we sum the series using common tricks for power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n(2n-1)} = -z \text{hypergeom}([1, 1, \frac{1}{2}], \frac{3}{2}, [2, 2], -z)$$

Info.

*not_given*

Comment.

*no_comment*

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3.5 Problem.

Sum the following power series

\[ \sum_{n=0}^{\infty} \frac{(2n + 1)z^n}{n!} \]

Hint.

*prepare_a_combination_of_exponentials*

Solution.

We denote the \( n \)-th term in the series by

\[ a(n) = \frac{(2n + 1)z^n}{n!} \]

For the ratio test we need the term \( a(n + 1) \)

\[ a(n + 1) = \frac{(2n + 3)z^{(n+1)}}{(n + 1)!} \]

Ratio test computes

\[ \lim_{n \to \infty} \left| \frac{a(n+1)}{a(n)} \right| \]

and obtains in our case

\[ \lim_{n \to \infty} \left| \frac{(2n + 3)z^{(n+1)} n!}{(n + 1)! (2n + 1) z^n} \right| = 0 \]

Moreover we can check the root test computing

\[ \lim_{n \to \infty} \sqrt[n]{|a(n)|} \]

and obtain in our case

\[ \lim_{n \to \infty} \sqrt[n]{\left| \frac{(2n + 1)z^n}{n!} \right|^{\frac{1}{n}}} = \left| \lim_{n \to \infty} \left( \frac{(2n + 1)z^n}{n!} \right)^{\frac{1}{n}} \right| \]

From this we conclude the radius of convergence \( R \). For \(|z| < R\) we sum the series using common tricks for power series

\[ \sum_{n=0}^{\infty} \frac{(2n + 1)z^n}{n!} = 2e^z (\frac{1}{2} + z) \]

Info.

*not_given*

Comment.

*no_comment*
3.6 Problem.

Sum the following power series

\[ \sum_{n=0}^{\infty} \frac{(n^2 - 2) z^n}{2^n n!} \]

Hint.

*prepare a combination of exponentials*

Solution.

We denote the \( n \)-th term in the series by

\[ a(n) = \frac{(n^2 - 2) z^n}{2^n n!} \]

For the ratio test we need the term \( a(n+1) \)

\[ a(n+1) = \frac{((n+1)^2 - 2) z^{(n+1)}}{2^{(n+1)} (n+1)!} \]

Ratio test computes

\[ \lim_{n \to \infty} \left| \frac{a(n+1)}{a(n)} \right| \]

and obtains in our case

\[ 2^n \left| \frac{((n+1)^2 - 2) z^{(n+1)}}{2^{(n+1)} (n+1)!} \frac{(n^2 - 2) z^n}{2^n n!} \right| = 0 \]

Moreover we can check the root test computing

\[ \lim_{n \to \infty} \sqrt[n]{|a(n)|} \]

and obtain in our case

\[ \lim_{n \to \infty} \frac{(n^2 - 2) z^n}{2^n n!} \left( \frac{1}{2} \right)^n = \left| \lim_{n \to \infty} \left( \frac{(n^2 - 2) z^n}{2^n n!} \right)^{\frac{1}{n}} \right| \]

From this we conclude the radius of convergence \( R \). For \( |z| < R \) we sum the series using common tricks for power series

\[ \sum_{n=0}^{\infty} \frac{(n^2 - 2) z^n}{2^n n!} = -2 e^{(1/2 z)} + \frac{1}{2} \sqrt{2} e^{(1/2 z)} z + \frac{1}{2} \left( \sqrt{2} + 1 \right) e^{(1/2 z)} z^2 + \frac{1}{4} e^{(1/2 z)} z^2 \]

Info.

*not given*

Comment.

*no comment*
3.7 Problem.

Sum the following power series

\[ \sum_{n=0}^{\infty} n(n+1)z^n \]

Hint.

\textit{divide by }z\textit{ and integrate twice}

Solution.

We denote the \(n\)-th term in the series by

\[ a(n) = n(n+1)z^n \]

For the ratio test we need the term \(a(n+1)\)

\[ a(n+1) = (n+1)(n+2)z^{n+1} \]

Ratio test computes

\[ \lim_{n \to \infty} \left| \frac{a(n+1)}{a(n)} \right| \]

and obtains in our case

\[ \lim_{n \to \infty} \left| \frac{(n+2)z^{n+1}}{n} \right| = |z| \]

Moreover we can check the root test computing

\[ \lim_{n \to \infty} \sqrt[n]{|a(n)|} \]

and obtain in our case

\[ \lim_{n \to \infty} [(n+1)z^n]^{\frac{1}{n}} = |z| \]

From this we conclude the radius of convergence \(R\). For \(|z| < R\) we sum the series using common tricks for power series

\[ \sum_{n=0}^{\infty} n(n+1)z^n = -2 \frac{z}{(z-1)^3} \]

Info.

\textit{not given}

Comment.

\textit{no_comment}
3.8 Problem.

Sum the following power series

$$\sum_{n=0}^{\infty} \frac{n z^n}{5^n}$$

Hint.

divide_by_z_and_integrate

Solution.

We denote the $n$-th term in the series by

$$a(n) = \frac{n z^n}{5^n}$$

For the ratio test we need the term $a(n + 1)$

$$a(n + 1) = \frac{(n + 1) z^{n+1}}{5^{n+1}}$$

Ratio test computes

$$\lim_{n \to \infty} \frac{|a(n + 1)|}{|a(n)|}$$

and obtains in our case

$$\frac{5^n}{n} \left| \frac{(n + 1) z^{n+1}}{5^{n+1}} \right| = \frac{1}{5} |z|$$

Moreover we can check the root test computing

$$\lim_{n \to \infty} \sqrt[n]{|a(n)|}$$

and obtain in our case

$$\lim_{n \to \infty} \sqrt[n]{\frac{n z^n}{5^n}} = \lim_{n \to \infty} \left( \frac{n z^n}{5^n} \right)^{\frac{1}{n}}$$

From this we conclude the radius of convergence $R$. For $|z| < R$ we sum the series using common tricks for power series

$$\sum_{n=0}^{\infty} \frac{n z^n}{5^n} = 5 \frac{z}{(z - 5)^2}$$

Info.

not_given

Comment.

no_comment

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3.9 Problem.

Sum the following power series

$$\sum_{n=1}^{\infty} n^2 z^{(n-1)}$$

Hint.

integrate, divide by $z$ and integrate

Solution.

We denote the $n$-th term in the series by

$$a(n) = n^2 z^{(n-1)}$$

For the ratio test we need the term $a(n+1)$

$$a(n+1) = (n+1)^2 z^n$$

Ratio test computes

$$\lim_{n \to \infty} \left| \frac{a(n+1)}{a(n)} \right|$$

and obtains in our case

$$\lim_{n \to \infty} \left| \frac{(n+1)^2 z^n}{n^2 z^{(n-1)}} \right| = |z|$$

Moreover we can check the root test computing

$$\lim_{n \to \infty} \sqrt[n]{|a(n)|}$$

and obtain in our case

$$\lim_{n \to \infty} \sqrt[n]{|n^2 z^{(n-1)}|} = |z|$$

From this we conclude the radius of convergence $R$. For $|z| < R$ we sum the series using common tricks for power series

$$\sum_{n=1}^{\infty} n^2 z^{(n-1)} = \frac{1 + z}{(-z + 1)^3}$$

Info.

not given

Comment.

no comment
3.10 Problem.

Sum the following power series

\[ \sum_{n=0}^{\infty} \frac{z^{(2n)}}{(2n)!} \]

Hint.

*manipulate to exponentials*

Solution.

We denote the \( n \)-th term in the series by

\[ a(n) = \frac{z^{(2n)}}{(2n)!} \]

For the ratio test we need the term \( a(n + 1) \)

\[ a(n + 1) = \frac{z^{(2n+2)}}{(2n + 2)!} \]

Ratio test computes

\[ \lim_{n \to \infty} \frac{|a(n + 1)|}{|a(n)|} \]

and obtains in our case

\[ \lim_{n \to \infty} \left| \frac{\frac{z^{(2n+2)}}{(2n + 2)!} \cdot \frac{z^{(2n)}}{(2n)!}}{\frac{z^{(2n)}}{(2n)!} \cdot \frac{z^{(2n+2)}}{(2n + 2)!}} \right| = 0 \]

Moreover we can check the root test computing

\[ \lim_{n \to \infty} \sqrt[n]{|a(n)|} \]

and obtain in our case

\[ \lim_{n \to \infty} \left( \frac{\frac{z^{(2n)}}{(2n)!}}{\frac{z^{(2n+2)}}{(2n + 2)!}} \right)^{\frac{1}{n}} = 0 \]

From this we conclude the radius of convergence \( R \). For \( |z| < R \) we sum the series using common tricks for power series

\[ \sum_{n=0}^{\infty} \frac{z^{(2n)}}{(2n)!} = \cosh(z) \]

Info.

*not given*

Comment.

*no comment*
3.11 Problem.

Sum the following power series
\[ \sum_{n=0}^{\infty} \frac{z^{(2n-1)}}{(2n-1)!} \]

Hint.

*manipulate to exponentials*

Solution.

We denote the \(n\)-th term in the series by
\[ a(n) = \frac{z^{(2n-1)}}{(2n-1)!} \]

For the ratio test we need the term \(a(n+1)\)
\[ a(n+1) = \frac{z^{(2n+1)}}{(2n+1)!} \]

Ratio test computes
\[ \lim_{n \to \infty} \left| \frac{a(n+1)}{a(n)} \right| \]

and obtains in our case
\[ \lim_{n \to \infty} \left| \frac{z^{(2n+1)}}{(2n+1)!} \cdot \frac{(2n-1)!}{z^{(2n-1)}} \right| = 0 \]

Moreover we can check the root test computing
\[ \lim_{n \to \infty} \sqrt[n]{|a(n)|} \]

and obtain in our case
\[ \lim_{n \to \infty} \sqrt[n]{|a(n)|} \]

From this we conclude the radius of convergence \(R\). For \(|z| < R\) we sum the series using common tricks for power series
\[ \sum_{n=0}^{\infty} \frac{z^{(2n-1)}}{(2n-1)!} = \sinh(z) \]

Info.

*not given*

Comment.

*no comment*
3.12 Problem.

Sum the following power series

\[ \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)} \]

Hint.

*multiple_by_z_and_derive_twice*

Solution.

We denote the \( n \)-th term in the series by

\[ a(n) = \frac{z^n}{n(n+1)} \]

For the ratio test we need the term \( a(n+1) \)

\[ a(n+1) = \frac{z^{n+1}}{(n+1)(n+2)} \]

Ratio test computes

\[ \lim_{n \to \infty} \frac{|a(n+1)|}{|a(n)|} \]

and obtains in our case

\[ \lim_{n \to \infty} \left| \frac{z^{n+1} n}{(n+2) z^n} \right| = |z| \]

Moreover we can check the root test computing

\[ \lim_{n \to \infty} \sqrt[n]{|a(n)|} \]

and obtain in our case

\[ \lim_{n \to \infty} \left| \frac{z^n}{n(n+1)} \right|^{1/n} = |z| \]

From this we conclude the radius of convergence \( R \). For \( |z| < R \) we sum the series using common tricks for power series

\[ \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)} = \frac{1}{2} z \left( 2 \left( \frac{\ln(-z+1)}{z} - 1 \right) (z-1) - \frac{-2z+2}{z-1} \right) \]

Info.

*not_given*

Comment.

*no_comment*
3.13 Problem.
Sum the following power series
\[ \sum_{n=0}^{\infty} \frac{z^n}{n!} \]

Hint.

*manipulate_to_exponentials*

Solution.
We denote the \( n \)-th term in the series by
\[ a(n) = \frac{z^n}{n!} \]
For the ratio test we need the term \( a(n+1) \)
\[ a(n + 1) = \frac{z^{(n+1)}}{(n+1)!} \]
Ratio test computes
\[ \lim_{n \to \infty} \frac{|a(n+1)|}{|a(n)|} \]
and obtains in our case
\[ \lim_{n \to \infty} \left| \frac{z^{(n+1)}}{(n+1)!} \right| z^n = 0 \]
Moreover we can check the root test computing
\[ \lim_{n \to \infty} \sqrt[n]{|a(n)|} \]
and obtain in our case
\[ \lim_{n \to \infty} \left| \frac{z^n}{n!} \right| \left( \frac{z^n}{n!} \right)^{\frac{1}{n}} = 0 \]
From this we conclude the radius of convergence \( R \). For \( |z| < R \) we sum the series using common tricks for power series
\[ \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z \]
Info.

*not_given*

Comment.

*no_comment*

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